Hilbert-Schmidt Geometry of n-Level Jakóbczyk-Siennicki Two-Dimensional Quantum Systems

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Abstract

Jakóbczyk and Siennicki studied *two*-dimensional sections of a set of (generalized) Bloch vectors corresponding to $n \times n$ density matrices of two-qubit systems (that is, the case $n = 4$). They found essentially five different types of (nontrivial) separability regimes. We compute the Euclidean/Hilbert-Schmidt (HS) separability *probabilities* assigned to these regimes, and conduct parallel *two*-dimensional sectional analyses for the *higher*-level cases $n = 6, 8, 9$ and 10. Making use of the newly-introduced capability for integration over *implicitly* defined regions of version 5.1 of Mathematica — as we have also fruitfully done in the $n = 4$ *three*-parameter entropy-maximization-based study quant-ph/0507203 — we obtain a wide-ranging variety of *exact* HS-probabilities. For $n > 6$, the probabilities are those of having a *partial positive transpose* (PPT). For the $n = 6$ case, we also obtain *biseparability* probabilities; in the $n = 8, 9$ instances, bi-PPT probabilities; and for $n = 8$, *tri*-PPT probabilities. By far, the most frequently recorded probability for $n > 4$ is $\frac{\pi}{4} \approx 0.785398$. We also conduct a number of related analyses, pertaining to the (one-dimensional) *boundaries* (both exterior and interior) of the separability and PPT domains, and attempt (with quite limited success) some *exact* calculations pertaining to the *9*-dimensional (real) and *15*-dimensional (complex) convex sets of two-qubit density matrices — for which exact HS-separability probabilities have been conjectured, but not yet verified.

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I. INTRODUCTION

There has been considerable recent interest [1, 2, 3] in understanding how one can, from the spherical coordinate point-of view, generalize to n-level quantum systems ($n \geq 2$) the familiar Bloch ball representation of the two-level quantum systems ($n = 2$) — in which the pure states form the bounding spherical surface ("Bloch sphere") of the unit ball in three-dimensional Euclidean space. Kimura and Kossakowski have expressed the generalized Bloch representation of an $n \times n$ density matrix in the form [1, eq. (3)]

$$\rho = \frac{\mathrm{tr}\rho}{n} I_n + \frac{1}{2} \Sigma_{i=1}^{n^2-1}(\mathrm{tr}\rho\lambda_i)\lambda_i, \tag{1}$$

where I_n is the identity operator, and the λ_i's are the $(n^2 - 1)$ orthogonal generators of $SU(n)$, forming a basis of the set of all the linear operators with respect to the Hilbert-Schmidt inner product.

An interesting application of these concepts was made by Jakóbczyk and Siennicki (JS) [4]. They examined all those two-qubit ($n = 4$) systems describable as *two*-dimensional sections of sets of (generalized) Bloch (coherence [3]) vectors. (The *totality* of 4×4 density matrices, on the other hand, comprises a *fifteen*-dimensional convex set — the $n \times n$ density matrices being $(n^2 - 1)$-dimensional in nature — so thirteen of the fifteen $SU(4)$ orthogonal generators [Gell-mann matrices] are assigned null weight in the JS $n = 4$ analyses. That is, thirteen of the fifteen coefficients, $(\mathrm{tr}\lambda_i)$ in the expansion (1) are zero.)

Since there were only two parameters involved in each of their scenarios, JS were able to present *planar* diagrams depicting the feasible regions, as well as those subsets of these regions composed of separable states. In their Fig. 1, JS exhibited thirteen possible types of parameter domains. Further, in their Fig. 2, they showed six different (nontrivial) separability scenarios (two of which — labelled "EF)" and "FE)" by JS – are simply geometric reflections of one another).

We will, firstly (sec. II), in this study, evaluate the sizes (areas) of these six (two-dimensional) domains and nontrivial subdomains, in terms of the *Hilbert-Schmidt* (HS) metric — a task JS did not explicitly address. (The HS-distance between two density operators ρ_1, ρ_2 is defined as $\sqrt{\mathrm{Tr}(\rho_1 - \rho_2)^2}$ [5, eq. (2.3)].) Then, we extend the JS analyses to the cases $n = 6$ (sec. III), 8 (sec. IV), 9 (sec. V) and 10 (sec. VI), in which various *multi*partite — as opposed to simply *bi*partite scenarios can arise. For all these instances,

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except $n = 10$, we additionally obtain the HS-lengths of the (one-dimensional) boundary states (that is, those with *degenerate* spectra) and the HS-probabilities that states lying on this boundary are separable. (Motivated by our extensive numerical results given in [6] and [7], Szarek, Bengtsson and Życzkowski have recently proved "that the probability to find a random state to be separable equals 2 times the probability to find a random boundary state to be separable, provided the random states are generated uniformly with respect to the Hilbert-Schmidt (Euclidean) distance" [8]). Also, we compute in certain cases, the HS-lengths of the (interior) boundaries dividing one domain of interest from another. (The interior states generically have *nondegenerate* spectra.) Then (sec. VII), we undertake some analyses involving *three* (rather than two) parameters. These prove to be much more *problematical* in nature (cf. [9]).

We also report, at the end (sec. VII C), some initial steps in an attempt to determine exact *upper* bounds for the HS-volumes of the *separable 9*-dimensional real and *15*-dimensional complex 4×4 density matrices. (Only in these computations — in order to compare our formulas with known HS-volumes of separable and nonseparable steps [5] — do we not take the HS-volume element to be *unity*.)

Our computations in this paper were *greatly* facilitated by a new feature of the pro-gramming language Mathematica (version 5.1) — the capacity to integrate over *implicitly* defined regions. (This feature was also employed by us in [9], in a somewhat related two-qubit context, in which the Jaynes maximum-entropy principle was employed.) We, first, found explicit forms for the n eigenvalues of the various $n \times n$ matrices *and* for their par-tial transposes. Then we required, in the several integrations, using the new feature (thus, saving us from the laborious task of having to specify large numbers of particular integra-tion limits and do corresponding detailed bookkeeping), simply that these eigenvalues be *nonnegative*. This ensured that we either had, in fact, the requisite *density* matrices and/or *positive partial transposes* (PPT) of density matrices.

II. THE QUBIT-QUBIT CASE $n = 4$ OF JAKØBCZYK AND SIENNICKI

For the (geometrically-reflected) scenarios that JS labeled "EF)" and "FE)", we have found (Table I) that the Hilbert-Schmidt volume (cf. [5]) of separable *and* nonseparable states is $\frac{2\sqrt{2}}{3}$ and of the separable states *alone* is $\frac{2}{3}$. So the corresponding separability

probability (taking ratios) is — elegantly — $\frac{1}{\sqrt{2}} \approx 0.707107$.

For the scenario "CK)", possessing a triangular separability domain, the total volume is $\frac{4\sqrt{\frac{2}{3}}}{3}$ and the separability probability is $\frac{1}{24}(9 + 2\sqrt{3}\pi) \approx 0.828450$. For "GH')", the total volume is $\frac{9}{32}\sqrt{\frac{3}{2}}\pi$ and the separability probability, $\frac{26\sqrt{2}+27\tan^{-1}(2\sqrt{2})}{27\pi} \approx 0.825312$. For "KC)", the total volume is $\frac{\sqrt{2}\pi}{3}$ and the separability probability is (the smallest) $\frac{1}{3} + \frac{\sqrt{3}}{2\pi} \approx 0.608998$. For "HG')", the total HS-volume is $\frac{3}{2\sqrt{2}}$ and the HS-separability probability is (the largest of the five) $\frac{52+27\sqrt{2}\sec^{-1}(3)}{48\sqrt{6}} \approx 0.842035$.

TABLE I:

JS scenario	HS total vol.	HS separable vol.	HS sep. prob.	num. approx.
EF) and FE)	$\frac{2\sqrt{2}}{3}$	$\frac{2}{3}$	$\frac{1}{\sqrt{2}}$	0.707107
CK)	$\frac{4\sqrt{\frac{2}{3}}}{3}$	$\frac{1}{18}\left(3\sqrt{6} + 2\sqrt{2}\pi\right)$	$\frac{9+2\sqrt{3}\pi}{24}$	0.828450
GH')	$\frac{9}{32}\sqrt{\frac{3}{2}}\pi$	$\frac{1}{192}\left(52\sqrt{3} + 27\sqrt{6}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right)$	$\frac{26\sqrt{2}+27\tan^{-1}(2\sqrt{2})}{27\pi}$	0.825312
HG')	$\frac{3}{2\sqrt{2}}$	$\frac{1}{192}\left(52\sqrt{3} + 27\sqrt{6}\sec^{-1}(3)\right)$	$\frac{52+27\sqrt{2}\sec^{-1}(3)}{48\sqrt{6}}$	0.842035
KC)	$\frac{\sqrt{2}\pi}{3}$	$\frac{1}{18}\left(3\sqrt{6} + 2\sqrt{2}\pi\right)$	$\frac{1}{3} + \frac{\sqrt{3}}{2\pi}$	0.608998

Now, let us present again most of these results (Table I) in the form of the array (2). We do so because we will also present all the results of our subsequent analyses below (for $n > 4$) in this manner (which we have found to be the most convenient for directly incorporating our large-scale Mathematica computer-generated analyses into this report).

In the first column of (2) are given the identifying numbers of a *pair* of Gell-Mann matrices (generators of $SU(4)$) — which, in fact, can be seen to fully agree with the numbering (and associated scenario-labelling) of JS [4, p. 389]. (Here and further, we will *always* adhere to the conventional/standard numbering [10, sec. III] of the Lie generators of $SU(n)$, so that our results should be reproducible/verifiable to others. We list the pairs in lexographic order, using the first pair as the representative for its equivalence class.) In the second column of (2) are shown the *number* of distinct unordered pairs of $SU(4)$ generators which share the same total (separable and nonseparable) HS volume, as well as the same separable HS volume, and consequently, identical HS separability probabilities. The third column gives us these HS total volumes, the fourth column, the HS separability probabilities and the last (fifth) column, numerical approximations to the exact probabilities (which, of course, we see — being probabilities — do not exceed the value 1). (Due to space/page width constraints, we

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were unable to generally present in these data arrays the HS separable volumes too, though they can, of course, be deduced from the total volume and the separability probability.)

$$
\begin{pmatrix}
\{3,6\} & 4 & \frac{2\sqrt{2}}{3} & \frac{1}{\sqrt{2}} & & 0.707107 \\
\{6,8\} & 2 & \frac{9}{32}\sqrt{\frac{3}{2}}\pi & \frac{26\sqrt{2}+27\tan^{-1}\left(2\sqrt{2}\right)}{27\pi} & 0.825312 \\
\{6,15\} & 2 & \frac{4\sqrt{\frac{2}{3}}}{3} & \frac{1}{24}\left(9+2\sqrt{3}\pi\right) & 0.828450 \\
\{8,9\} & 2 & \frac{3}{2\sqrt{2}} & \frac{52+27\sqrt{2}\sec^{-1}(3)}{48\sqrt{6}} & 0.842035 \\
\{9,15\} & 2 & \frac{\sqrt{2}\pi}{3} & \frac{1}{3}+\frac{\sqrt{3}}{2\pi} & 0.608998
\end{pmatrix}.
\tag{2}
$$

Thus, twelve of the $210 = 15 \cdot 14$ possible unordered pairs of Gell-Mann matrices are associated with nontrivial (< 1) separability probabilities [4, p. 389].

A. Boundary states

For the scenario associated with the pair of Gell-Mann matrices $\{3,6\}$, the HS-length of the (one-dimensional) *boundary* states (that is, those with *degenerate* spectra) is $\frac{3}{2}$, and of the bounding states which are separable, $\frac{1}{2}$. For the pair $\{6,8\}$, the analogous results are $\frac{3}{2\sqrt{2}} \approx 1.06066$ and 1; for $\{6,15\}$, they are $\frac{2}{3}$ and $\frac{1}{2}$; and for $\{8,9\}$, $\frac{3\sqrt{3}}{4}$ and $\frac{\sqrt{3}}{2}$, for a separability probability of boundary states of $\frac{2}{3}$. For the last $\{9,15\}$ of the five scenarios, we have $\frac{2}{\sqrt{3}}$ and 1. Let us now present these results in the following array form (which we will adopt for our more extensive results further below):

$$
\begin{pmatrix}
\{3,6\} & \frac{3}{2} & \frac{1}{2} & \frac{1}{3} & 0.333333 \\
\{6,8\} & \frac{3}{2\sqrt{2}} & 1 & \frac{2\sqrt{2}}{3} & 0.942809 \\
\{6,15\} & \frac{2}{3} & \frac{1}{2} & \frac{3}{4} & 0.75 \\
\{8,9\} & \frac{3\sqrt{3}}{4} & \frac{\sqrt{3}}{2} & \frac{2}{3} & 0.666667 \\
\{9,15\} & \frac{2}{\sqrt{3}} & 1 & \frac{\sqrt{3}}{2} & 0.866025
\end{pmatrix}.
\tag{3}
$$

B. Length of Separability-Nonseparability *Interior* Boundary

In the following array, we present the HS-length of the common border separating the nonseparable (entangled) states from the separable ones. The states lying along this interior border generically have *nondegenerate* spectra.

$$
\begin{pmatrix}
\{3,6\} & \{6,8\} & \{6,15\} & \{8,9\} & \{9,15\} \\
\frac{1}{2} & 1 & 1 & \frac{\sqrt{3}}{4} & \frac{1}{2}
\end{pmatrix}.
\tag{4}
$$

III. THE QUBIT-QUTRIT CASE $n = 6$

A. 3×2 Decomposition

Moving on from the $n = 4$ case specifically studied by Jacóbczyk and Siennicki to $n = 6$ (cf. [6]), we compute the partial transposes of the 6×6 density matrices, corresponding to two-dimensional sections of the set of Bloch vectors. We, first, transpose in place the (2^2) four 3×3 blocks of the density matrices. By the Peres-Horodecki criterion, such density matrices with *positive* partial transposes must be *separable*.

We obtained the following results, presented in the same manner as (2).

$$
\begin{pmatrix}
\{1, 13\} & 48 & \frac{4}{9} & \frac{\pi}{4} & 0.785398 \\
\{3, 11\} & 4 & \frac{8\sqrt{2}}{27} & \frac{1}{\sqrt{2}} & 0.707107 \\
\{3, 13\} & 4 & \frac{4}{9} & \frac{5}{6} & 0.833333 \\
\{3, 25\} & 4 & \frac{8\sqrt{2}}{27} & \frac{5}{4\sqrt{2}} & 0.883883 \\
\{8, 13\} & 4 & \frac{2}{3} & \frac{1}{\sqrt{3}} & 0.577350 \\
\{8, 25\} & 4 & \frac{\sqrt{2}}{3} & \sqrt{\frac{2}{3}} & 0.816497 \\
\{11, 15\} & 4 & \frac{4\sqrt{2}\pi}{27} & \frac{1}{3} + \frac{3\sqrt{3}}{4\pi} & 0.746830 \\
\{11, 24\} & 2 & \frac{25\sqrt{\frac{5}{2}}}{72} & \frac{2}{5} + \frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right) & 0.863648 \\
\{13, 24\} & 2 & \frac{25\sqrt{\frac{5}{2}}}{72} & \frac{8}{75}\left(-2 + 5\sqrt{5}\right) & 0.979236 \\
\{13, 35\} & 4 & \frac{4\sqrt{\frac{3}{5}}}{5} & \frac{1}{12}\left(5 + 3\sqrt{5}\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)\right) & 0.886838 \\
\{15, 16\} & 4 & \frac{32\sqrt{2}}{81} & \frac{1}{32}\left(9\sqrt{3} + 4\pi\right) & 0.879838 \\
\{16, 24\} & 2 & \frac{25}{144}\sqrt{\frac{5}{2}}\pi & \frac{4 + 5\sin^{-1}\left(\frac{4}{5}\right)}{5\pi} & 0.549815 \\
\{20, 24\} & 2 & \frac{25}{144}\sqrt{\frac{5}{2}}\pi & \frac{92 + 75\sin^{-1}\left(\frac{4}{5}\right)}{75\pi} & 0.685627 \\
\{24, 25\} & 2 & \frac{25}{27\sqrt{2}} & 1 - \frac{2}{5\sqrt{5}} & 0.821115 \\
\{24, 27\} & 2 & \frac{25}{27\sqrt{2}} & \frac{92 + 75\cos^{-1}\left(\frac{3}{5}\right)}{80\sqrt{5}} & 0.903076 \\
\{25, 35\} & 4 & \frac{\sqrt{3}\pi}{5} & \frac{\sqrt{5} + 3\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)}{3\pi} & 0.504975
\end{pmatrix}. \tag{5}
$$

1. Boundary states

In the following array, we list, first the scenario pair, then, the HS-length of the boundary states (those with degenerate spectra), then, the HS-length of those boundary states which

are separable, then, the separability probability and a numerical approximation to it.

$$
\begin{pmatrix}
\{1,13\} & \frac{2}{3} & 0 & 0 & 0. \\
\{3,11\} & 1 & \frac{1}{3} & \frac{1}{3} & 0.333333 \\
\{3,13\} & \frac{2}{3} & \frac{1}{3} & \frac{1}{2} & 0.5 \\
\{3,25\} & 1 & \frac{1}{3} & \frac{1}{3} & 0.333333 \\
\{8,13\} & \frac{2}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{4} & 0.25 \\
\{8,25\} & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}} & \frac{2}{3} & 0.666667 \\
\{11,15\} & \frac{4}{3\sqrt{3}} & \frac{2}{3} & \frac{\sqrt{3}}{2} & 0.866025 \\
\{11,24\} & \frac{5}{12} & \frac{1}{3} & \frac{4}{5} & 0.8 \\
\{13,24\} & \frac{5}{6} & \frac{\sqrt{5}}{3} & \frac{2}{\sqrt{5}} & 0.894427 \\
\{13,35\} & \frac{2}{5} & \frac{1}{3} & \frac{5}{6} & 0.833333 \\
\{15,16\} & \frac{4\sqrt{\frac{2}{3}}}{3} & \sqrt{\frac{2}{3}} & \frac{3}{4} & 0.75 \\
\{16,24\} & \frac{5}{6} & \frac{2}{3} & \frac{4}{5} & 0.8 \\
\{20,24\} & \frac{5}{6} & \frac{2}{3} & \frac{4}{5} & 0.8 \\
\{24,25\} & \frac{5\sqrt{\frac{5}{2}}}{6} & \frac{\sqrt{\frac{5}{2}}}{6} & \frac{1}{5} & 0.2 \\
\{24,27\} & \frac{5\sqrt{\frac{5}{2}}}{6} & \frac{\sqrt{10}}{3} & \frac{4}{5} & 0.8 \\
\{25,35\} & \frac{2}{\sqrt{5}} & \frac{2}{3} & \frac{\sqrt{5}}{3} & 0.745356
\end{pmatrix}. \tag{6}
$$

2. Length of Separability-Nonseparability Interior Boundary

In the following arrays, we present the HS-length of the common border separating the nonseparable (entangled) states from the separable ones for each specific scenario.

$$
\begin{pmatrix}
\{1,13\} & \{3,11\} & \{3,13\} & \{3,25\} & \{8,13\} & \{8,25\} & \{11,15\} & \{11,24\} \\
\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{3} & \frac{2}{3}
\end{pmatrix} \tag{7}
$$

$$
\begin{pmatrix}
\{13,24\} & \{13,35\} & \{15,16\} & \{16,24\} & \{20,24\} & \{24,25\} & \{24,27\} & \{25,35\} \\
\frac{\sqrt{5}}{3} & \frac{2}{3} & \frac{\sqrt{\frac{2}{3}}}{3} & \frac{1}{3} & \frac{2}{3} & \frac{\sqrt{10}}{3} & \frac{\sqrt{\frac{5}{2}}}{6} & \frac{1}{3}
\end{pmatrix}.
$$

B. 2×3 Decomposition

Here, we compute the partial transposes of the same collection of 6×6 density matrices, corresponding to two-dimensional sections of the set of Bloch vectors, by transposing in

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place the (3^2) nine 2×2 blocks of the density matrices — rather than the four $(2^2)3 \times 3$ blocks as previously (sec. III A).

We obtained the following results.

$$
\begin{pmatrix}
\{3,6\} & 8 & \frac{8\sqrt{2}}{27} & \frac{1}{\sqrt{2}} & 0.707107 \\[4pt]
\{4,18\} & 48 & \frac{4}{9} & \frac{\pi}{4} & 0.785398 \\[4pt]
\{6,8\} & 2 & \frac{1}{8}\sqrt{\frac{3}{2}}\pi & \frac{26\sqrt{2}+27\tan^{-1}\left(2\sqrt{2}\right)}{27\pi} & 0.825312 \\[4pt]
\{6,15\} & 2 & \frac{16\sqrt{\frac{2}{3}}}{27} & \frac{1}{24}\left(9+2\sqrt{3}\pi\right) & 0.828450 \\[4pt]
\{8,9\} & 2 & \frac{\sqrt{2}}{3} & \frac{52+27\sqrt{2}\sec^{-1}(3)}{48\sqrt{6}} & 0.842035 \\[4pt]
\{8,22\} & 2 & \frac{1}{\sqrt{3}} & \frac{8}{9} & 0.888889 \\[4pt]
\{8,29\} & 2 & \frac{2}{3} & \frac{4}{3\sqrt{3}} & 0.769800 \\[4pt]
\{9,15\} & 2 & \frac{4\sqrt{2}\pi}{27} & \frac{1}{3}+\frac{\sqrt{3}}{2\pi} & 0.608998 \\[4pt]
\{15,22\} & 2 & \frac{32\sqrt{\frac{2}{3}}}{27} & \frac{1}{2} & 0.500000 \\[4pt]
\{15,29\} & 2 & \frac{32\sqrt{2}}{81} & \frac{\sqrt{3}}{2} & 0.866025 \\[4pt]
\{18,24\} & 4 & \frac{25}{144}\sqrt{\frac{5}{2}}\pi & \frac{92+75\sin^{-1}\left(\frac{4}{5}\right)}{75\pi} & 0.685627 \\[4pt]
\{18,35\} & 4 & \frac{4\sqrt{\frac{3}{5}}}{5} & \frac{1}{12}\left(5+3\sqrt{5}\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)\right) & 0.886838 \\[4pt]
\{24,25\} & 4 & \frac{25}{27\sqrt{2}} & \frac{92+75\cos^{-1}\left(\frac{3}{5}\right)}{80\sqrt{5}} & 0.903076 \\[4pt]
\{25,35\} & 4 & \frac{\sqrt{3}\pi}{5} & \frac{\sqrt{5}+3\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)}{3\pi} & 0.504975
\end{pmatrix}. \tag{8}
$$

There are now only fourteen rows, while in the preceding qubit-qutrit analysis (5) there were sixteen. In both analyses, though, there are 48 unordered pairs of Lie generators which yield HS separability probabilities equal to $\frac{\pi}{4}$.

Here, we again present the results, restricting consideration to the boundary (degenerate spectra) states, in the same form as previously (3).

$$
\begin{pmatrix}
\{3,6\} & 1 & \frac{1}{3} & \frac{1}{3} & 0.333333 \\
\{4,18\} & \frac{2}{3} & 0 & 0 & 0. \\
\{6,8\} & \frac{1}{\sqrt{2}} & \frac{2}{3} & \frac{2\sqrt{2}}{3} & 0.942809 \\
\{6,15\} & \frac{4}{9} & \frac{1}{3} & \frac{3}{4} & 0.75 \\
\{8,9\} & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}} & \frac{2}{3} & 0.666667 \\
\{8,22\} & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}} & \frac{2}{3} & 0.666667 \\
\{8,29\} & \frac{2}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{4} & 0.25 \\
\{9,15\} & \frac{4}{3\sqrt{3}} & \frac{2}{3} & \frac{\sqrt{3}}{2} & 0.866025 \\
\{15,22\} & \frac{5\sqrt{\frac{2}{3}}}{3} & \frac{\sqrt{\frac{2}{3}}}{3} & \frac{1}{5} & 0.2 \\
\{15,29\} & \frac{4\sqrt{\frac{2}{3}}}{3} & \sqrt{\frac{2}{3}} & \frac{3}{4} & 0.75 \\
\{18,24\} & \frac{5}{6} & \frac{2}{3} & \frac{4}{5} & 0.8 \\
\{18,35\} & \frac{2}{5} & \frac{1}{3} & \frac{5}{6} & 0.833333 \\
\{24,25\} & \frac{5\sqrt{\frac{5}{2}}}{6} & \frac{\sqrt{10}}{3} & \frac{4}{5} & 0.8 \\
\{25,35\} & \frac{2}{\sqrt{5}} & \frac{2}{3} & \frac{\sqrt{5}}{3} & 0.745356
\end{pmatrix}
\tag{9}
$$

2. *Length of Separability-Nonseparability Interior Boundary*

In the following arrays, we present the HS-length of the common border separating the nonseparable (entangled) states from the separable ones for each specific scenario.

$$
\begin{pmatrix}
\{3,6\} & \{4,18\} & \{6,8\} & \{6,15\} & \{8,9\} & \{8,22\} & \{8,29\} \\
\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{pmatrix}
\tag{10}
$$

$$
\begin{pmatrix}
\{9,15\} & \{15,22\} & \{15,29\} & \{18,24\} & \{18,35\} & \{24,25\} & \{25,35\} \\
\frac{1}{3} & \sqrt{\frac{2}{3}} & \frac{\sqrt{\frac{2}{3}}}{3} & \frac{2}{3} & \frac{2}{3} & \frac{\sqrt{\frac{5}{2}}}{6} & \frac{1}{3}
\end{pmatrix} .
$$

C. Biseparable HS probabilities

Now, we determine which of the two-dimensional set of 6×6 density matrices have positive partial transposes for *both* forms of partial transposition used in sec. III A III B. The results

we obtained were:

$$
\begin{pmatrix}
\{1,13\} & 88 & \frac{4}{9} & \frac{\pi}{4} & 0.785398 \\[6pt]
\{3,6\} & 12 & \frac{8\sqrt{2}}{27} & \frac{1}{\sqrt{2}} & 0.707107 \\[6pt]
\{3,13\} & 4 & \frac{4}{9} & \frac{5}{6} & 0.833333 \\[6pt]
\{3,27\} & 2 & \frac{8\sqrt{2}}{27} & \frac{5}{4\sqrt{2}} & 0.883883 \\[6pt]
\{6,8\} & 2 & \frac{1}{8}\sqrt{\frac{3}{2}}\pi & \frac{26\sqrt{2}+27\tan^{-1}\left(2\sqrt{2}\right)}{27\pi} & 0.825312 \\[6pt]
\{6,15\} & 2 & \frac{16\sqrt{\frac{2}{3}}}{27} & \frac{1}{24}\left(9+2\sqrt{3}\pi\right) & 0.828450 \\[6pt]
\{8,9\} & 2 & \frac{\sqrt{2}}{3} & \frac{52+27\sqrt{2}\sec^{-1}(3)}{48\sqrt{6}} & 0.842035 \\[6pt]
\{8,13\} & 4 & \frac{2}{3} & \frac{1}{\sqrt{3}} & 0.577350 \\[6pt]
\{8,22\} & 2 & \frac{1}{\sqrt{3}} & \frac{8}{9} & 0.888889 \\[6pt]
\{8,25\} & 4 & \frac{\sqrt{2}}{3} & \sqrt{\frac{2}{3}} & 0.816497 \\[6pt]
\{8,29\} & 2 & \frac{2}{3} & \frac{4}{3\sqrt{3}} & 0.769800 \\[6pt]
\{9,15\} & 2 & \frac{4\sqrt{2}\pi}{27} & \frac{1}{3}+\frac{\sqrt{3}}{2\pi} & 0.608998 \\[6pt]
\{11,15\} & 4 & \frac{4\sqrt{2}\pi}{27} & \frac{1}{3}+\frac{3\sqrt{3}}{4\pi} & 0.746830 \\[6pt]
\{11,24\} & 2 & \frac{25\sqrt{\frac{5}{2}}}{72} & \frac{2}{5}+\frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right) & 0.863648 \\[6pt]
\{13,24\} & 2 & \frac{25\sqrt{\frac{5}{2}}}{72} & \frac{8}{75}\left(-2+5\sqrt{5}\right) & 0.979236 \\[6pt]
\{13,35\} & 8 & \frac{4\sqrt{\frac{3}{5}}}{5} & \frac{1}{12}\left(5+3\sqrt{5}\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)\right) & 0.886838 \\[6pt]
\{15,16\} & 4 & \frac{32\sqrt{2}}{81} & \frac{1}{32}\left(9\sqrt{3}+4\pi\right) & 0.879838 \\[6pt]
\{15,22\} & 2 & \frac{32\sqrt{\frac{2}{3}}}{27} & \frac{1}{2} & 0.500000 \\[6pt]
\{15,29\} & 2 & \frac{32\sqrt{2}}{81} & \frac{\sqrt{3}}{2} & 0.866025 \\[6pt]
\{16,24\} & 2 & \frac{25}{144}\sqrt{\frac{5}{2}}\pi & \frac{4+5\sin^{-1}\left(\frac{4}{5}\right)}{5\pi} & 0.549815 \\[6pt]
\{18,24\} & 6 & \frac{25}{144}\sqrt{\frac{5}{2}}\pi & \frac{92+75\sin^{-1}\left(\frac{4}{5}\right)}{75\pi} & 0.685627 \\[6pt]
\{24,25\} & 2 & \frac{25}{27\sqrt{2}} & \frac{3\left(4+5\cos^{-1}\left(\frac{3}{5}\right)\right)}{16\sqrt{5}} & 0.724191 \\[6pt]
\{24,27\} & 4 & \frac{25}{27\sqrt{2}} & \frac{92+75\cos^{-1}\left(\frac{3}{5}\right)}{80\sqrt{5}} & 0.903076 \\[6pt]
\{25,35\} & 6 & \frac{\sqrt{3}\pi}{5} & \frac{\sqrt{5}+3\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)}{3\pi} & 0.504975
\end{pmatrix} . \qquad (11)
$$

1. Boundary States

Concerning the corresponding one-dimensional (exterior) boundary (degenerate spectra) states we found:

$$
\begin{pmatrix}
\{1,13\} & \frac{2}{3} & 0 & 0 & 0. \\
\{3,6\} & 1 & \frac{1}{3} & \frac{1}{3} & 0.333333 \\
\{3,13\} & \frac{2}{3} & \frac{1}{3} & \frac{1}{2} & 0.5 \\
\{3,27\} & 1 & \frac{1}{3} & \frac{1}{3} & 0.333333 \\
\{6,8\} & \frac{1}{\sqrt{2}} & \frac{2}{3} & \frac{2\sqrt{2}}{3} & 0.942809 \\
\{6,15\} & \frac{4}{9} & \frac{1}{3} & \frac{3}{4} & 0.75 \\
\{8,9\} & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}} & \frac{2}{3} & 0.666667 \\
\{8,13\} & \frac{2}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{4} & 0.25 \\
\{8,22\} & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}} & \frac{2}{3} & 0.666667 \\
\{8,25\} & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}} & \frac{2}{3} & 0.666667 \\
\{8,29\} & \frac{2}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{4} & 0.25 \\
\{9,15\} & \frac{4}{3\sqrt{3}} & \frac{2}{3} & \frac{\sqrt{3}}{2} & 0.866025 \\
\{11,15\} & \frac{4}{3\sqrt{3}} & \frac{2}{3} & \frac{\sqrt{3}}{2} & 0.866025 \\
\{11,24\} & \frac{5}{12} & \frac{1}{3} & \frac{4}{5} & 0.8 \\
\{13,24\} & \frac{5}{6} & \frac{\sqrt{5}}{3} & \frac{2}{\sqrt{5}} & 0.894427 \\
\{13,35\} & \frac{2}{5} & \frac{1}{3} & \frac{5}{6} & 0.833333 \\
\{15,16\} & \frac{4\sqrt{\frac{2}{3}}}{3} & \sqrt{\frac{2}{3}} & \frac{3}{4} & 0.75 \\
\{15,22\} & \frac{5\sqrt{\frac{2}{3}}}{3} & \frac{\sqrt{\frac{2}{3}}}{3} & \frac{1}{5} & 0.2 \\
\{15,29\} & \frac{4\sqrt{\frac{2}{3}}}{3} & \sqrt{\frac{2}{3}} & \frac{3}{4} & 0.75 \\
\{16,24\} & \frac{5}{6} & \frac{2}{3} & \frac{4}{5} & 0.8 \\
\{18,24\} & \frac{5}{6} & \frac{2}{3} & \frac{4}{5} & 0.8 \\
\{24,25\} & \frac{5\sqrt{\frac{5}{2}}}{6} & 0 & 0 & 0. \\
\{24,27\} & \frac{5\sqrt{\frac{5}{2}}}{6} & \frac{\sqrt{10}}{3} & \frac{4}{5} & 0.8 \\
\{25,35\} & \frac{2}{\sqrt{5}} & \frac{2}{3} & \frac{\sqrt{5}}{3} & 0.745356
\end{pmatrix} . \tag{12}
$$

2. Length of Biseparability-Nonbiseparability Interior Boundary

In the following array, we present the HS-length of the common border separating the (generically nondegenerate) biseparable states from the non-biseparable ones for each specific scenario.

$$
\begin{pmatrix}
\{1,13\} & \frac{2}{3} \\
\{3,6\} & \frac{2}{3} \\
\{3,13\} & \frac{2}{3} \\
\{3,27\} & \frac{2}{3} \\
\{6,8\} & \frac{4}{3} \\
\{6,15\} & 1 \\
\{8,9\} & \frac{\sqrt{3}}{2} \\
\{8,13\} & \frac{\sqrt{3}}{2} \\
\{8,22\} & \frac{\sqrt{3}}{2} \\
\{8,25\} & \frac{\sqrt{3}}{2} \\
\{8,29\} & \frac{\sqrt{3}}{2} \\
\{9,15\} & 1 \\
\{11,15\} & \frac{4}{3} \\
\{11,24\} & 1 \\
\{13,24\} & \frac{2\sqrt{5}}{3} \\
\{13,35\} & 1 \\
\{15,16\} & \frac{4\sqrt{\frac{2}{3}}}{3} \\
\{15,22\} & \frac{4\sqrt{\frac{2}{3}}}{3} \\
\{15,29\} & \frac{4\sqrt{\frac{2}{3}}}{3} \\
\{16,24\} & 1 \\
\{18,24\} & \frac{4}{3} \\
\{24,25\} & \frac{\sqrt{\frac{5}{2}}}{6} + \frac{\sqrt{10}}{3} \\
\{24,27\} & \frac{\sqrt{\frac{5}{2}}}{6} + \frac{\sqrt{10}}{3} \\
\{25,35\} & \frac{2}{3}
\end{pmatrix} . \tag{13}
$$

IV. THE CASE $n = 8$

A. 4×2 Decomposition

Here, we compute the partial transposes of the 8×8 density matrices, corresponding to two-dimensional sections of the set of Bloch vectors, by, first, transposing in place the (2^2)

four 4×4 blocks of the density matrices. The results are

$$
\begin{pmatrix}
\{1,20\} & 192 & \frac{1}{4} & \frac{\pi}{4} & 0.785398 \\[4pt]
\{3,18\} & 4 & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & 0.707107 \\[4pt]
\{3,20\} & 8 & \frac{1}{4} & \frac{5}{6} & 0.833333 \\[4pt]
\{3,36\} & 8 & \frac{1}{3\sqrt{2}} & \frac{5}{4\sqrt{2}} & 0.883883 \\[4pt]
\{8,20\} & 4 & \frac{3}{8} & \frac{1}{\sqrt{3}} & 0.577350 \\[4pt]
\{8,22\} & 4 & \frac{3\sqrt{3}}{16} & \frac{7}{9} & 0.777778 \\[4pt]
\{8,36\} & 4 & \frac{3}{8\sqrt{2}} & \sqrt{\frac{2}{3}} & 0.816497 \\[4pt]
\{8,42\} & 2 & \frac{3\sqrt{3}}{16} & \frac{8}{9} & 0.888889 \\[4pt]
\{8,49\} & 4 & \frac{3}{8\sqrt{2}} & \frac{7}{3\sqrt{6}} & 0.952579 \\[4pt]
\{8,53\} & 2 & \frac{3}{8} & \frac{4}{3\sqrt{3}} & 0.769800 \\[4pt]
\{15,22\} & 6 & \frac{2\sqrt{\frac{2}{3}}}{3} & \frac{1}{2} & 0.500000 \\[4pt]
\{15,49\} & 6 & \frac{2\sqrt{2}}{9} & \frac{\sqrt{3}}{2} & 0.866025 \\[4pt]
\{18,24\} & 6 & \frac{25}{256}\sqrt{\frac{5}{2}}\pi & \frac{92+75\sin^{-1}\left(\frac{4}{5}\right)}{75\pi} & 0.685627 \\[4pt]
\{18,35\} & 2 & \frac{9\sqrt{\frac{3}{5}}}{20} & \frac{1}{12}\left(5+3\sqrt{5}\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)\right) & 0.886838 \\[4pt]
\{20,35\} & 4 & \frac{9\sqrt{\frac{3}{5}}}{20} & \frac{1}{108}\left(-25+24\sqrt{30}\right) & 0.985680 \\[4pt]
\{20,48\} & 4 & \frac{49\sqrt{\frac{7}{3}}}{192} & \frac{1}{28}\left(12+7\sqrt{6}\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)\right) & 0.903278 \\[4pt]
\{22,48\} & 4 & \frac{49\sqrt{\frac{7}{3}}}{192} & \frac{4}{147}\left(-9+7\sqrt{42}\right) & 0.989529 \\[4pt]
\{22,63\} & 6 & \frac{8}{7\sqrt{7}} & \frac{1}{16}\left(7+4\sqrt{7}\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)\right) & 0.915544 \\[4pt]
\{24,25\} & 6 & \frac{25}{48\sqrt{2}} & \frac{92+75\cos^{-1}\left(\frac{3}{5}\right)}{80\sqrt{5}} & 0.903076 \\[4pt]
\{25,35\} & 2 & \frac{9\sqrt{3}\pi}{80} & \frac{\sqrt{5}+3\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)}{3\pi} & 0.504975 \\[4pt]
\{29,35\} & 4 & \frac{9\sqrt{3}\pi}{80} & \frac{14\sqrt{5}+27\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)}{27\pi} & 0.636783 \\[4pt]
\{35,36\} & 4 & \frac{3}{5\sqrt{2}} & 1-\frac{5\sqrt{\frac{5}{6}}}{24} & 0.809819 \\[4pt]
\{35,38\} & 4 & \frac{3}{5\sqrt{2}} & \frac{14\sqrt{5}+27\cos^{-1}\left(\frac{2}{3}\right)}{24\sqrt{6}} & 0.918793 \\[4pt]
\{36,48\} & 4 & \frac{49}{384}\sqrt{\frac{7}{2}}\pi & \frac{2\sqrt{6}+7\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)}{7\pi} & 0.469522 \\[4pt]
\{42,48\} & 2 & \frac{49}{384}\sqrt{\frac{7}{2}}\pi & \frac{22\sqrt{6}+49\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)}{49\pi} & 0.596820 \\[4pt]
\{48,49\} & 4 & \frac{49}{72\sqrt{2}} & 1-\frac{3\sqrt{\frac{3}{14}}}{7} & 0.801610 \\[4pt]
\{48,53\} & 2 & \frac{49}{72\sqrt{2}} & \frac{3\left(22\sqrt{6}+49\cos^{-1}\left(\frac{5}{7}\right)\right)}{112\sqrt{7}} & 0.930129 \\[4pt]
\{49,63\} & 6 & \frac{2\pi}{7} & \frac{\sqrt{7}+4\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)}{4\pi} & 0.440596
\end{pmatrix} . \tag{14}
$$

1. Boundary states

The results pertaining to the *one*-dimensional (exterior) boundary (generically degenerate) states were:

$$
\begin{pmatrix}
\{1,20\} & \frac{1}{2} & 0 & 0 & 0. \\[4pt]
\{3,18\} & \frac{3}{4} & \frac{1}{4} & \frac{1}{3} & 0.333333 \\[4pt]
\{3,20\} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & 0.5 \\[4pt]
\{3,36\} & \frac{3}{4} & \frac{1}{4} & \frac{1}{3} & 0.333333 \\[4pt]
\{8,20\} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{8} & \frac{1}{4} & 0.25 \\[4pt]
\{8,22\} & \frac{3\sqrt{3}}{8} & \frac{\sqrt{3}}{8} & \frac{1}{3} & 0.333333 \\[4pt]
\{8,36\} & \frac{3\sqrt{3}}{8} & \frac{\sqrt{3}}{4} & \frac{2}{3} & 0.666667 \\[4pt]
\{8,42\} & \frac{3\sqrt{3}}{8} & \frac{\sqrt{3}}{4} & \frac{2}{3} & 0.666667 \\[4pt]
\{8,49\} & \frac{3\sqrt{3}}{8} & \frac{\sqrt{3}}{4} & \frac{2}{3} & 0.666667 \\[4pt]
\{8,53\} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{8} & \frac{1}{4} & 0.25 \\[4pt]
\{15,22\} & \frac{5}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{5} & 0.2 \\[4pt]
\{15,49\} & \sqrt{\frac{2}{3}} & \frac{\sqrt{\frac{3}{2}}}{2} & \frac{3}{4} & 0.75 \\[4pt]
\{18,24\} & \frac{5}{8} & \frac{1}{2} & \frac{4}{5} & 0.8 \\[4pt]
\{18,35\} & \frac{3}{10} & \frac{1}{4} & \frac{5}{6} & 0.833333 \\[4pt]
\{20,35\} & \frac{3}{5} & \sqrt{\frac{3}{10}} & \sqrt{\frac{5}{6}} & 0.912871 \\[4pt]
\{20,48\} & \frac{7}{24} & \frac{1}{4} & \frac{6}{7} & 0.857143 \\[4pt]
\{22,48\} & \frac{7}{12} & \frac{\sqrt{\frac{7}{6}}}{2} & \sqrt{\frac{6}{7}} & 0.92582 \\[4pt]
\{22,63\} & \frac{2}{7} & \frac{1}{4} & \frac{7}{8} & 0.875 \\[4pt]
\{24,25\} & \frac{5\sqrt{\frac{5}{2}}}{8} & \frac{\sqrt{\frac{5}{2}}}{2} & \frac{4}{5} & 0.8 \\[4pt]
\{25,35\} & \frac{3}{2\sqrt{5}} & \frac{1}{2} & \frac{\sqrt{5}}{3} & 0.745356 \\[4pt]
\{29,35\} & \frac{3}{2\sqrt{5}} & \frac{1}{2} & \frac{\sqrt{5}}{3} & 0.745356 \\[4pt]
\{35,36\} & \frac{3\sqrt{\frac{3}{5}}}{2} & \frac{\sqrt{\frac{3}{5}}}{4} & \frac{1}{6} & 0.166667 \\[4pt]
\{35,38\} & \frac{3\sqrt{\frac{3}{5}}}{2} & \frac{\sqrt{15}}{4} & \frac{5}{6} & 0.833333 \\[4pt]
\{36,48\} & \frac{7}{4\sqrt{6}} & \frac{1}{2} & \frac{2\sqrt{6}}{7} & 0.699854 \\[4pt]
\{42,48\} & \frac{7}{4\sqrt{6}} & \frac{1}{2} & \frac{2\sqrt{6}}{7} & 0.699854 \\[4pt]
\{48,49\} & \frac{7\sqrt{\frac{7}{3}}}{8} & \frac{\sqrt{\frac{7}{3}}}{8} & \frac{1}{7} & 0.142857 \\[4pt]
\{48,53\} & \frac{7\sqrt{\frac{7}{3}}}{8} & \frac{\sqrt{21}}{4} & \frac{6}{7} & 0.857143 \\[4pt]
\{49,63\} & \frac{2}{\sqrt{7}} & \frac{1}{2} & \frac{\sqrt{7}}{4} & 0.661438
\end{pmatrix} . \tag{15}
$$

Here, we see the appearance of (fully entangled) domains that have no separable component, at all.

B. 2×4 Decomposition

Now, we compute the partial transposes of the 8×8 density matrices, corresponding to two-dimensional sections of the set of Bloch vectors, by transposing in place the (4^4) sixteen 2×2 blocks of the density matrices. We obtained the following results.

$$
\begin{pmatrix}
\{3,6\} & 12 & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & 0.707107 \\[4pt]
\{4,18\} & 192 & \frac{1}{4} & \frac{\pi}{4} & 0.785398 \\[4pt]
\{6,8\} & 2 & \frac{9}{128}\sqrt{\frac{3}{2}}\pi & \frac{26\sqrt{2}+27\tan^{-1}\left(2\sqrt{2}\right)}{27\pi} & 0.825312 \\[4pt]
\{6,15\} & 2 & \frac{\sqrt{\frac{2}{3}}}{3} & \frac{1}{24}\left(9+2\sqrt{3}\pi\right) & 0.828450 \\[4pt]
\{8,9\} & 2 & \frac{3}{8\sqrt{2}} & \frac{52+27\sqrt{2}\sec^{-1}(3)}{48\sqrt{6}} & 0.842035 \\[4pt]
\{8,22\} & 4 & \frac{3\sqrt{3}}{16} & \frac{8}{9} & 0.888889 \\[4pt]
\{8,29\} & 4 & \frac{3}{8} & \frac{4}{3\sqrt{3}} & 0.769800 \\[4pt]
\{9,15\} & 2 & \frac{\pi}{6\sqrt{2}} & \frac{1}{3}+\frac{\sqrt{3}}{2\pi} & 0.608998 \\[4pt]
\{15,22\} & 4 & \frac{2\sqrt{\frac{2}{3}}}{3} & \frac{1}{2} & 0.500000 \\[4pt]
\{15,29\} & 4 & \frac{2\sqrt{2}}{9} & \frac{\sqrt{3}}{2} & 0.866025 \\[4pt]
\{18,24\} & 4 & \frac{25}{256}\sqrt{\frac{5}{2}}\pi & \frac{92+75\sin^{-1}\left(\frac{4}{5}\right)}{75\pi} & 0.685627 \\[4pt]
\{18,35\} & 4 & \frac{9\sqrt{\frac{3}{5}}}{20} & \frac{1}{12}\left(5+3\sqrt{5}\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)\right) & 0.886838 \\[4pt]
\{24,25\} & 4 & \frac{25}{48\sqrt{2}} & \frac{92+75\cos^{-1}\left(\frac{3}{5}\right)}{80\sqrt{5}} & 0.903076 \\[4pt]
\{24,46\} & 2 & \frac{5\sqrt{\frac{5}{2}}}{16} & \frac{14}{15} & 0.933333 \\[4pt]
\{24,57\} & 2 & \frac{25}{24\sqrt{2}} & \frac{7}{5\sqrt{5}} & 0.626099 \\[4pt]
\{25,35\} & 4 & \frac{9\sqrt{3}\pi}{80} & \frac{\sqrt{5}+3\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)}{3\pi} & 0.504975 \\[4pt]
\{35,46\} & 2 & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{6}} & 0.408248 \\[4pt]
\{35,57\} & 2 & \frac{3}{5\sqrt{2}} & \sqrt{\frac{5}{6}} & 0.912871 \\[4pt]
\{38,48\} & 6 & \frac{49}{384}\sqrt{\frac{7}{2}}\pi & \frac{22\sqrt{6}+49\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)}{49\pi} & 0.596820 \\[4pt]
\{38,63\} & 6 & \frac{8}{7\sqrt{7}} & \frac{1}{16}\left(7+4\sqrt{7}\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)\right) & 0.915544 \\[4pt]
\{48,49\} & 6 & \frac{49}{72\sqrt{2}} & \frac{3\left(22\sqrt{6}+49\cos^{-1}\left(\frac{5}{7}\right)\right)}{112\sqrt{7}} & 0.930129 \\[4pt]
\{49,63\} & 6 & \frac{2\pi}{7} & \frac{\sqrt{7}+4\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)}{4\pi} & 0.440596
\end{pmatrix}. \tag{16}
$$

We see that there are fewer rows in (16) than in (14), obtained by the alternative form of partial transposition.

1. *Boundary states*

Our analysis of the HS-*lengths* of the corresponding boundary states yielded:

$$
\begin{pmatrix}
\{3,6\} & \frac{3}{4} & \frac{1}{4} & \frac{1}{3} & 0.333333 \\
\{4,18\} & \frac{1}{2} & 0 & 0 & 0. \\
\{6,8\} & \frac{3}{4\sqrt{2}} & \frac{1}{2} & \frac{2\sqrt{2}}{3} & 0.942809 \\
\{6,15\} & \frac{1}{3} & \frac{1}{4} & \frac{3}{4} & 0.75 \\
\{8,9\} & \frac{3\sqrt{3}}{8} & \frac{\sqrt{3}}{4} & \frac{2}{3} & 0.666667 \\
\{8,22\} & \frac{3\sqrt{3}}{8} & \frac{\sqrt{3}}{4} & \frac{2}{3} & 0.666667 \\
\{8,29\} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{8} & \frac{1}{4} & 0.25 \\
\{9,15\} & \frac{1}{\sqrt{3}} & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0.866025 \\
\{15,22\} & \frac{5}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{5} & 0.2 \\
\{15,29\} & \sqrt{\frac{2}{3}} & \frac{\sqrt{\frac{3}{2}}}{2} & \frac{3}{4} & 0.75 \\
\{18,24\} & \frac{5}{8} & \frac{1}{2} & \frac{4}{5} & 0.8 \\
\{18,35\} & \frac{3}{10} & \frac{1}{4} & \frac{5}{6} & 0.833333 \\
\{24,25\} & \frac{5\sqrt{\frac{5}{2}}}{8} & \frac{\sqrt{\frac{5}{2}}}{2} & \frac{4}{5} & 0.8 \\
\{24,46\} & \frac{5\sqrt{\frac{5}{2}}}{8} & \frac{\sqrt{\frac{5}{2}}}{2} & \frac{4}{5} & 0.8 \\
\{24,57\} & \frac{3\sqrt{\frac{5}{2}}}{4} & \frac{\sqrt{\frac{5}{2}}}{8} & \frac{1}{6} & 0.166667 \\
\{25,35\} & \frac{3}{2\sqrt{5}} & \frac{1}{2} & \frac{\sqrt{5}}{3} & 0.745356 \\
\{35,46\} & \frac{7\sqrt{\frac{3}{5}}}{4} & \frac{\sqrt{\frac{3}{5}}}{4} & \frac{1}{7} & 0.142857 \\
\{35,57\} & \frac{3\sqrt{\frac{3}{5}}}{2} & \frac{\sqrt{15}}{4} & \frac{5}{6} & 0.833333 \\
\{38,48\} & \frac{7}{4\sqrt{6}} & \frac{1}{2} & \frac{2\sqrt{6}}{7} & 0.699854 \\
\{38,63\} & \frac{2}{7} & \frac{1}{4} & \frac{7}{8} & 0.875 \\
\{48,49\} & \frac{7\sqrt{\frac{7}{3}}}{8} & \frac{\sqrt{21}}{4} & \frac{6}{7} & 0.857143 \\
\{49,63\} & \frac{2}{\sqrt{7}} & \frac{1}{2} & \frac{\sqrt{7}}{4} & 0.661438
\end{pmatrix}. \tag{17}
$$

C. Bi-PPT

Here, we obtain the probabilities that an 8×8 density matrix will have a positive partial transpose, under *both* forms of partial transposition employed immediately above. Our results were:

$$
\begin{pmatrix}
\{1,20\} & 288 & \frac{1}{4} & \frac{\pi}{4} & 0.785398 \\
\{3,6\} & 12 & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & 0.707107 \\
\{3,20\} & 8 & \frac{1}{4} & \frac{5}{6} & 0.833333 \\
\{3,36\} & 4 & \frac{1}{3\sqrt{2}} & \frac{5}{4\sqrt{2}} & 0.883883 \\
\{6,8\} & 2 & \frac{9}{128}\sqrt{\frac{3}{2}}\pi & \frac{26\sqrt{2}+27\tan^{-1}\left(2\sqrt{2}\right)}{27\pi} & 0.825312 \\
\{6,15\} & 2 & \frac{\sqrt{\frac{2}{3}}}{3} & \frac{1}{24}\left(9+2\sqrt{3}\pi\right) & 0.828450 \\
\{8,9\} & 2 & \frac{3}{8\sqrt{2}} & \frac{52+27\sqrt{2}\sec^{-1}(3)}{48\sqrt{6}} & 0.842035 \\
\{8,20\} & 4 & \frac{3}{8} & \frac{1}{\sqrt{3}} & 0.577350 \\
\{8,22\} & 2 & \frac{3\sqrt{3}}{16} & \frac{2}{3} & 0.666667 \\
\{8,31\} & 2 & \frac{3\sqrt{3}}{16} & \frac{7}{9} & 0.777778 \\
\{8,36\} & 4 & \frac{3}{8\sqrt{2}} & \sqrt{\frac{2}{3}} & 0.816497 \\
\{8,42\} & 2 & \frac{3\sqrt{3}}{16} & \frac{8}{9} & 0.888889 \\
\{8,49\} & 4 & \frac{3}{8\sqrt{2}} & \frac{7}{3\sqrt{6}} & 0.952579 \\
\{8,53\} & 2 & \frac{3}{8} & \frac{4}{3\sqrt{3}} & 0.769800 \\
\{9,15\} & 2 & \frac{\pi}{6\sqrt{2}} & \frac{1}{3}+\frac{\sqrt{3}}{2\pi} & 0.608998 \\
\{15,22\} & 6 & \frac{2\sqrt{\frac{2}{3}}}{3} & \frac{1}{2} & 0.500000 \\
\{15,29\} & 8 & \frac{2\sqrt{2}}{9} & \frac{\sqrt{3}}{2} & 0.866025 \\
\{18,24\} & 6 & \frac{25}{256}\sqrt{\frac{5}{2}}\pi & \frac{92+75\sin^{-1}\left(\frac{4}{5}\right)}{75\pi} & 0.685627 \\
\{18,35\} & 4 & \frac{9\sqrt{\frac{3}{5}}}{20} & \frac{1}{12}\left(5+3\sqrt{5}\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)\right) & 0.886838
\end{pmatrix}
\tag{18}
$$

$$
\begin{pmatrix}
\{20,35\} & 2 & \frac{9\sqrt{\frac{3}{5}}}{20} & \frac{1}{108}\left(-25+24\sqrt{30}\right) & 0.985680 \\[2mm]
\{20,48\} & 4 & \frac{49\sqrt{\frac{7}{3}}}{192} & \frac{1}{28}\left(12+7\sqrt{6}\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)\right) & 0.903278 \\[2mm]
\{22,48\} & 4 & \frac{49\sqrt{\frac{7}{3}}}{192} & \frac{4}{147}\left(-9+7\sqrt{42}\right) & 0.989529 \\[2mm]
\{22,63\} & 10 & \frac{8}{7\sqrt{7}} & \frac{1}{16}\left(7+4\sqrt{7}\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)\right) & 0.915544 \\[2mm]
\{24,25\} & 8 & \frac{25}{48\sqrt{2}} & \frac{92+75\cos^{-1}\left(\frac{3}{5}\right)}{80\sqrt{5}} & 0.903076 \\[2mm]
\{24,46\} & 2 & \frac{5\sqrt{\frac{5}{2}}}{16} & \frac{14}{15} & 0.933333 \\[2mm]
\{24,57\} & 2 & \frac{25}{24\sqrt{2}} & \frac{7}{5\sqrt{5}} & 0.626099 \\[2mm]
\{25,35\} & 4 & \frac{9\sqrt{3}\pi}{80} & \frac{\sqrt{5}+3\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)}{3\pi} & 0.504975 \\[2mm]
\{31,35\} & 2 & \frac{9\sqrt{3}\pi}{80} & \frac{14\sqrt{5}+27\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)}{27\pi} & 0.636783 \\[2mm]
\{35,36\} & 4 & \frac{3}{5\sqrt{2}} & 1-\frac{5\sqrt{\frac{5}{6}}}{24} & 0.809819 \\[2mm]
\{35,38\} & 4 & \frac{3}{5\sqrt{2}} & \frac{14\sqrt{5}+27\cos^{-1}\left(\frac{2}{3}\right)}{24\sqrt{6}} & 0.918793 \\[2mm]
\{35,46\} & 2 & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{6}} & 0.408248 \\[2mm]
\{35,57\} & 2 & \frac{3}{5\sqrt{2}} & \sqrt{\frac{5}{6}} & 0.912871 \\[2mm]
\{36,48\} & 4 & \frac{49}{384}\sqrt{\frac{7}{2}}\pi & \frac{2\sqrt{6}+7\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)}{7\pi} & 0.469522 \\[2mm]
\{42,48\} & 4 & \frac{49}{384}\sqrt{\frac{7}{2}}\pi & \frac{22\sqrt{6}+49\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)}{49\pi} & 0.596820 \\[2mm]
\{48,49\} & 2 & \frac{49}{72\sqrt{2}} & \frac{3\left(2\sqrt{6}+7\cos^{-1}\left(\frac{5}{7}\right)\right)}{16\sqrt{7}} & 0.731739 \\[2mm]
\{48,51\} & 2 & \frac{49}{72\sqrt{2}} & 1-\frac{3\sqrt{\frac{3}{14}}}{7} & 0.801610 \\[2mm]
\{48,53\} & 4 & \frac{49}{72\sqrt{2}} & \frac{3\left(22\sqrt{6}+49\cos^{-1}\left(\frac{5}{7}\right)\right)}{112\sqrt{7}} & 0.930129 \\[2mm]
\{49,63\} & 8 & \frac{2\pi}{7} & \frac{\sqrt{7}+4\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)}{4\pi} & 0.440596
\end{pmatrix}.
$$

1.   Boundary states

The lengths, separable lengths and separability probabilities of the corresponding (exterior/degenerate spectra) boundary states are given in the following array:

$$
\begin{pmatrix}
\{1,20\} & \frac{1}{2} & 0 & 0 & 0. \\
\{3,6\} & \frac{3}{4} & \frac{1}{4} & \frac{1}{3} & 0.333333 \\
\{3,20\} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & 0.5 \\
\{3,36\} & \frac{3}{4} & \frac{1}{4} & \frac{1}{3} & 0.333333 \\
\{6,8\} & \frac{3}{4\sqrt{2}} & \frac{1}{2} & \frac{2\sqrt{2}}{3} & 0.942809 \\
\{6,15\} & \frac{1}{3} & \frac{1}{4} & \frac{3}{4} & 0.75 \\
\{8,9\} & \frac{3\sqrt{3}}{8} & \frac{\sqrt{3}}{4} & \frac{2}{3} & 0.666667 \\
\{8,20\} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{8} & \frac{1}{4} & 0.25 \\
\{8,22\} & \frac{3\sqrt{3}}{8} & 0 & 0 & 0. \\
\{8,31\} & \frac{3\sqrt{3}}{8} & \frac{\sqrt{3}}{8} & \frac{1}{3} & 0.333333 \\
\{8,36\} & \frac{3\sqrt{3}}{8} & \frac{\sqrt{3}}{4} & \frac{2}{3} & 0.666667 \\
\{8,42\} & \frac{3\sqrt{3}}{8} & \frac{\sqrt{3}}{4} & \frac{2}{3} & 0.666667 \\
\{8,49\} & \frac{3\sqrt{3}}{8} & \frac{\sqrt{3}}{4} & \frac{2}{3} & 0.666667 \\
\{8,53\} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{8} & \frac{1}{4} & 0.25 \\
\{9,15\} & \frac{1}{\sqrt{3}} & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0.866025 \\
\{15,22\} & \frac{5}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{5} & 0.2 \\
\{15,29\} & \sqrt{\frac{2}{3}} & \frac{\sqrt{\frac{3}{2}}}{2} & \frac{3}{4} & 0.75 \\
\{18,24\} & \frac{5}{8} & \frac{1}{2} & \frac{4}{5} & 0.8 \\
\{18,35\} & \frac{3}{10} & \frac{1}{4} & \frac{5}{6} & 0.833333
\end{pmatrix}
\tag{19}
$$

$$
\begin{pmatrix}
\{20,35\} & \frac{3}{5} & \sqrt{\frac{3}{10}} & \sqrt{\frac{5}{6}} & 0.912871 \\
\{20,48\} & \frac{7}{24} & \frac{1}{4} & \frac{6}{7} & 0.857143 \\
\{22,48\} & \frac{7}{12} & \frac{\sqrt{7}}{2} & \sqrt{\frac{6}{7}} & 0.92582 \\
\{22,63\} & \frac{2}{7} & \frac{1}{4} & \frac{7}{8} & 0.875 \\
\{24,25\} & \frac{5\sqrt{\frac{5}{2}}}{8} & \frac{\sqrt{\frac{5}{2}}}{2} & \frac{4}{5} & 0.8 \\
\{24,46\} & \frac{5\sqrt{\frac{5}{2}}}{8} & \frac{\sqrt{\frac{5}{2}}}{2} & \frac{4}{5} & 0.8 \\
\{24,57\} & \frac{3\sqrt{\frac{5}{2}}}{4} & \frac{\sqrt{\frac{5}{2}}}{8} & \frac{1}{6} & 0.166667 \\
\{25,35\} & \frac{3}{2\sqrt{5}} & \frac{1}{2} & \frac{\sqrt{5}}{3} & 0.745356 \\
\{31,35\} & \frac{3}{2\sqrt{5}} & \frac{1}{2} & \frac{\sqrt{5}}{3} & 0.745356 \\
\{35,36\} & \frac{3\sqrt{\frac{3}{5}}}{2} & \frac{\sqrt{\frac{3}{5}}}{4} & \frac{1}{6} & 0.166667 \\
\{35,38\} & \frac{3\sqrt{\frac{3}{5}}}{2} & \frac{\sqrt{15}}{4} & \frac{5}{6} & 0.833333 \\
\{35,46\} & \frac{7\sqrt{\frac{3}{5}}}{4} & \frac{\sqrt{\frac{3}{5}}}{4} & \frac{1}{7} & 0.142857 \\
\{35,57\} & \frac{3\sqrt{\frac{3}{5}}}{2} & \frac{\sqrt{15}}{4} & \frac{5}{6} & 0.833333 \\
\{36,48\} & \frac{7}{4\sqrt{6}} & \frac{1}{2} & \frac{2\sqrt{6}}{7} & 0.699854 \\
\{42,48\} & \frac{7}{4\sqrt{6}} & \frac{1}{2} & \frac{2\sqrt{6}}{7} & 0.699854 \\
\{48,49\} & \frac{7\sqrt{\frac{7}{3}}}{8} & 0 & 0 & 0. \\
\{48,51\} & \frac{7\sqrt{\frac{7}{3}}}{8} & \frac{\sqrt{\frac{7}{3}}}{8} & \frac{1}{7} & 0.142857 \\
\{48,53\} & \frac{7\sqrt{\frac{7}{3}}}{8} & \frac{\sqrt{21}}{4} & \frac{6}{7} & 0.857143 \\
\{49,63\} & \frac{2}{\sqrt{7}} & \frac{1}{2} & \frac{\sqrt{7}}{4} & 0.661438
\end{pmatrix} .
$$

D. Tri-ppt

Now, we derive the probabilities that an 8×8 density matrix will have a positive partial transpose, not only under *both* forms of partial transposition previously employed, as in sec. IV C, but also under a *third* (independent) form obtained, first, applying a certain 8×8 *permutation* matrix ([11, eq. (3)]) to the original 8×8 density matrix, *then* transposing in

place the resultant four 4×4 blocks. We obtained the following results.

$$
\begin{pmatrix}
\{1, 20\} & 288 & \frac{1}{4} & \frac{\pi}{4} & 0.785398 \\[2mm]
\{3, 6\} & 12 & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & 0.707107 \\[2mm]
\{3, 20\} & 4 & \frac{1}{4} & \frac{5}{6} & 0.833333 \\[2mm]
\{3, 22\} & 4 & \frac{1}{4} & \frac{2}{3} & 0.666667 \\[2mm]
\{3, 36\} & 4 & \frac{1}{3\sqrt{2}} & \frac{5}{4\sqrt{2}} & 0.883883 \\[2mm]
\{6, 8\} & 2 & \frac{9}{128}\sqrt{\frac{3}{2}}\pi & \frac{26\sqrt{2}+27\tan^{-1}\left(2\sqrt{2}\right)}{27\pi} & 0.825312 \\[2mm]
\{6, 15\} & 2 & \frac{\sqrt{\frac{2}{3}}}{3} & \frac{1}{24}\left(9+2\sqrt{3}\pi\right) & 0.828450 \\[2mm]
\{8, 9\} & 2 & \frac{3}{8\sqrt{2}} & \frac{52+27\sqrt{2}\sec^{-1}(3)}{48\sqrt{6}} & 0.842035 \\[2mm]
\{8, 20\} & 4 & \frac{3}{8} & \frac{1}{\sqrt{3}} & 0.577350 \\[2mm]
\{8, 22\} & 2 & \frac{3\sqrt{3}}{16} & \frac{2}{3} & 0.666667 \\[2mm]
\{8, 31\} & 2 & \frac{3\sqrt{3}}{16} & \frac{7}{9} & 0.777778 \\[2mm]
\{8, 36\} & 6 & \frac{3}{8\sqrt{2}} & \sqrt{\frac{2}{3}} & 0.816497 \\[2mm]
\{8, 42\} & 2 & \frac{3\sqrt{3}}{16} & \frac{8}{9} & 0.888889 \\[2mm]
\{8, 51\} & 2 & \frac{3}{8\sqrt{2}} & \frac{7}{3\sqrt{6}} & 0.952579 \\[2mm]
\{8, 53\} & 2 & \frac{3}{8} & \frac{4}{3\sqrt{3}} & 0.769800 \\[2mm]
\{9, 15\} & 2 & \frac{\pi}{6\sqrt{2}} & \frac{1}{3}+\frac{\sqrt{3}}{2\pi} & 0.608998 \\[2mm]
\{15, 22\} & 6 & \frac{2\sqrt{\frac{2}{3}}}{3} & \frac{1}{2} & 0.500000 \\[2mm]
\{15, 29\} & 10 & \frac{2\sqrt{2}}{9} & \frac{\sqrt{3}}{2} & 0.866025 \\[2mm]
\{18, 24\} & 6 & \frac{25}{256}\sqrt{\frac{5}{2}}\pi & \frac{92+75\sin^{-1}\left(\frac{4}{5}\right)}{75\pi} & 0.685627 \\[2mm]
\{18, 35\} & 4 & \frac{9\sqrt{\frac{3}{5}}}{20} & \frac{1}{12}\left(5+3\sqrt{5}\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)\right) & 0.886838
\end{pmatrix}
\tag{20}
$$

$$
\begin{pmatrix}
\{20,35\} & 2 & \frac{9\sqrt{\frac{3}{5}}}{20} & \frac{1}{108}\left(-25+24\sqrt{30}\right) & 0.985680 \\[8pt]
\{20,48\} & 6 & \frac{49\sqrt{\frac{7}{3}}}{192} & \frac{1}{28}\left(12+7\sqrt{6}\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)\right) & 0.903278 \\[8pt]
\{22,63\} & 12 & \frac{8}{7\sqrt{7}} & \frac{1}{16}\left(7+4\sqrt{7}\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)\right) & 0.915544 \\[8pt]
\{24,25\} & 10 & \frac{25}{48\sqrt{2}} & \frac{92+75\cos^{-1}\left(\frac{3}{5}\right)}{80\sqrt{5}} & 0.903076 \\[8pt]
\{24,46\} & 2 & \frac{5\sqrt{\frac{5}{2}}}{16} & \frac{14}{15} & 0.933333 \\[8pt]
\{24,57\} & 2 & \frac{25}{24\sqrt{2}} & \frac{7}{5\sqrt{5}} & 0.626099 \\[8pt]
\{25,35\} & 4 & \frac{9\sqrt{3}\pi}{80} & \frac{\sqrt{5}+3\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)}{3\pi} & 0.504975 \\[8pt]
\{31,35\} & 2 & \frac{9\sqrt{3}\pi}{80} & \frac{14\sqrt{5}+27\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)}{27\pi} & 0.636783 \\[8pt]
\{31,48\} & 2 & \frac{49\sqrt{\frac{7}{3}}}{192} & \frac{4}{147}\left(-9+7\sqrt{42}\right) & 0.989529 \\[8pt]
\{35,36\} & 2 & \frac{3}{5\sqrt{2}} & 1-\frac{5\sqrt{\frac{5}{6}}}{24} & 0.809819 \\[8pt]
\{35,38\} & 4 & \frac{3}{5\sqrt{2}} & \frac{1}{8}\sqrt{\frac{3}{2}}\left(\sqrt{5}+3\cos^{-1}\left(\frac{2}{3}\right)\right) & 0.728612 \\[8pt]
\{35,46\} & 2 & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{6}} & 0.408248 \\[8pt]
\{35,51\} & 2 & \frac{3}{5\sqrt{2}} & \frac{14\sqrt{5}+27\cos^{-1}\left(\frac{2}{3}\right)}{24\sqrt{6}} & 0.918793 \\[8pt]
\{35,57\} & 2 & \frac{3}{5\sqrt{2}} & \sqrt{\frac{5}{6}} & 0.912871 \\[8pt]
\{36,48\} & 4 & \frac{49}{384}\sqrt{\frac{7}{2}}\pi & \frac{2\sqrt{6}+7\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)}{7\pi} & 0.469522 \\[8pt]
\{42,48\} & 4 & \frac{49}{384}\sqrt{\frac{7}{2}}\pi & \frac{22\sqrt{6}+49\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)}{49\pi} & 0.596820 \\[8pt]
\{48,49\} & 2 & \frac{49}{72\sqrt{2}} & \frac{3\left(2\sqrt{6}+7\cos^{-1}\left(\frac{5}{7}\right)\right)}{16\sqrt{7}} & 0.731739 \\[8pt]
\{48,51\} & 2 & \frac{49}{72\sqrt{2}} & 1-\frac{3\sqrt{\frac{3}{14}}}{7} & 0.801610 \\[8pt]
\{48,53\} & 4 & \frac{49}{72\sqrt{2}} & \frac{3\left(22\sqrt{6}+49\cos^{-1}\left(\frac{5}{7}\right)\right)}{112\sqrt{7}} & 0.930129 \\[8pt]
\{49,63\} & 8 & \frac{2\pi}{7} & \frac{\sqrt{7}+4\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)}{4\pi} & 0.440596
\end{pmatrix}.
$$

Here there are only four generator pairs yielding the probability $\frac{5}{6}$, while there were eight in simply the "Bi-PPT" case (sec. IV C).

1. Boundary States

Now, we obtained from the analysis of the one-dimensional (exterior) boundary states the results:

$$
\begin{pmatrix}
\{1,20\} & \frac{1}{2} & 0 & 0 & 0. \\
\{3,6\} & \frac{3}{4} & \frac{1}{4} & \frac{1}{3} & 0.333333 \\
\{3,20\} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & 0.5 \\
\{3,22\} & \frac{1}{2} & 0 & 0 & 0. \\
\{3,36\} & \frac{3}{4} & \frac{1}{4} & \frac{1}{3} & 0.333333 \\
\{6,8\} & \frac{3}{4\sqrt{2}} & \frac{1}{2} & \frac{2\sqrt{2}}{3} & 0.942809 \\
\{6,15\} & \frac{1}{3} & \frac{1}{4} & \frac{3}{4} & 0.75 \\
\{8,9\} & \frac{3\sqrt{3}}{8} & \frac{\sqrt{3}}{4} & \frac{2}{3} & 0.666667 \\
\{8,20\} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{8} & \frac{1}{4} & 0.25 \\
\{8,22\} & \frac{3\sqrt{3}}{8} & 0 & 0 & 0. \\
\{8,31\} & \frac{3\sqrt{3}}{8} & \frac{\sqrt{3}}{8} & \frac{1}{3} & 0.333333 \\
\{8,36\} & \frac{3\sqrt{3}}{8} & \frac{\sqrt{3}}{4} & \frac{2}{3} & 0.666667 \\
\{8,42\} & \frac{3\sqrt{3}}{8} & \frac{\sqrt{3}}{4} & \frac{2}{3} & 0.666667 \\
\{8,51\} & \frac{3\sqrt{3}}{8} & \frac{\sqrt{3}}{4} & \frac{2}{3} & 0.666667 \\
\{8,53\} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{8} & \frac{1}{4} & 0.25 \\
\{9,15\} & \frac{1}{\sqrt{3}} & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0.866025 \\
\{15,22\} & \frac{5}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{5} & 0.2 \\
\{15,29\} & \sqrt{\frac{2}{3}} & \frac{\sqrt{\frac{3}{2}}}{2} & \frac{3}{4} & 0.75 \\
\{18,24\} & \frac{5}{8} & \frac{1}{2} & \frac{4}{5} & 0.8 \\
\{18,35\} & \frac{3}{10} & \frac{1}{4} & \frac{5}{6} & 0.833333
\end{pmatrix}
\tag{21}
$$

$$
\begin{pmatrix}
\{20,35\} & \frac{3}{5} & \sqrt{\frac{3}{10}} & \sqrt{\frac{5}{6}} & 0.912871 \\
\{20,48\} & \frac{7}{24} & \frac{1}{4} & \frac{6}{7} & 0.857143 \\
\{22,63\} & \frac{2}{7} & \frac{1}{4} & \frac{7}{8} & 0.875 \\
\{24,25\} & \frac{5\sqrt{\frac{5}{2}}}{8} & \frac{\sqrt{\frac{5}{2}}}{2} & \frac{4}{5} & 0.8 \\
\{24,46\} & \frac{5\sqrt{\frac{5}{2}}}{8} & \frac{\sqrt{\frac{5}{2}}}{2} & \frac{4}{5} & 0.8 \\
\{24,57\} & \frac{3\sqrt{\frac{5}{2}}}{4} & \frac{\sqrt{\frac{5}{2}}}{8} & \frac{1}{6} & 0.166667 \\
\{25,35\} & \frac{3}{2\sqrt{5}} & \frac{1}{2} & \frac{\sqrt{5}}{3} & 0.745356 \\
\{31,35\} & \frac{3}{2\sqrt{5}} & \frac{1}{2} & \frac{\sqrt{5}}{3} & 0.745356 \\
\{31,48\} & \frac{7}{12} & \frac{\sqrt{\frac{7}{6}}}{2} & \sqrt{\frac{6}{7}} & 0.92582 \\
\{35,36\} & \frac{3\sqrt{\frac{3}{5}}}{2} & \frac{\sqrt{\frac{3}{5}}}{4} & \frac{1}{6} & 0.166667 \\
\{35,38\} & \frac{3\sqrt{\frac{3}{5}}}{2} & 0 & 0 & 0. \\
\{35,46\} & \frac{7\sqrt{\frac{3}{5}}}{4} & \frac{\sqrt{\frac{3}{5}}}{4} & \frac{1}{7} & 0.142857 \\
\{35,51\} & \frac{3\sqrt{\frac{3}{5}}}{2} & \frac{\sqrt{15}}{4} & \frac{5}{6} & 0.833333 \\
\{35,57\} & \frac{3\sqrt{\frac{3}{5}}}{2} & \frac{\sqrt{15}}{4} & \frac{5}{6} & 0.833333 \\
\{36,48\} & \frac{7}{4\sqrt{6}} & \frac{1}{2} & \frac{2\sqrt{6}}{7} & 0.699854 \\
\{42,48\} & \frac{7}{4\sqrt{6}} & \frac{1}{2} & \frac{2\sqrt{6}}{7} & 0.699854 \\
\{48,49\} & \frac{7\sqrt{\frac{7}{3}}}{8} & 0 & 0 & 0. \\
\{48,51\} & \frac{7\sqrt{\frac{7}{3}}}{8} & \frac{\sqrt{\frac{7}{3}}}{8} & \frac{1}{7} & 0.142857 \\
\{48,53\} & \frac{7\sqrt{\frac{7}{3}}}{8} & \frac{\sqrt{21}}{4} & \frac{6}{7} & 0.857143 \\
\{49,63\} & \frac{2}{\sqrt{7}} & \frac{1}{2} & \frac{\sqrt{7}}{4} & 0.661438
\end{pmatrix}
.
$$

2. Length of Triseparability-Nontriseparability Interior Boundary

In the following array, we present the HS-length of the common (interior) border separating the triseparable states from the non-triseparable ones for each specific scenario.

$$
\begin{pmatrix}
\{1,20\} & 1 \\
\{3,6\} & \frac{3}{4} \\
\{3,20\} & \frac{3}{4} \\
\{3,22\} & \frac{1}{2} \\
\{3,36\} & \frac{3}{4} \\
\{6,8\} & \frac{3}{2} \\
\{6,15\} & \frac{5}{4} \\
\{8,9\} & \frac{\sqrt{3}}{2} \\
\{8,20\} & \frac{5\sqrt{3}}{8} \\
\{8,22\} & \frac{5\sqrt{3}}{8} \\
\{8,31\} & \frac{5\sqrt{3}}{8} \\
\{8,36\} & \frac{\sqrt{3}}{2} \\
\{8,42\} & \frac{\sqrt{3}}{2} \\
\{8,51\} & \frac{\sqrt{3}}{2} \\
\{8,53\} & \frac{5\sqrt{3}}{8} \\
\{9,15\} & 1 \\
\{15,22\} & \frac{3\sqrt{\frac{3}{2}}}{2} \\
\{15,29\} & \frac{7}{2\sqrt{6}} \\
\{18,24\} & \frac{3}{2} \\
\{18,35\} & \frac{5}{4}
\end{pmatrix}
\tag{22}
$$

$$
\begin{pmatrix}
\{20,35\} & 3\sqrt{\frac{3}{10}} \\
\{20,48\} & \frac{5}{4} \\
\{22,63\} & 1 \\
\{24,25\} & \frac{3\sqrt{\frac{5}{2}}}{4} \\
\{24,46\} & \frac{3\sqrt{\frac{5}{2}}}{4} \\
\{24,57\} & \frac{9\sqrt{\frac{5}{2}}}{8} \\
\{25,35\} & 1 \\
\{31,35\} & \frac{3}{2} \\
\{31,48\} & \frac{\sqrt{\frac{21}{2}}}{2} \\
\{35,36\} & \frac{11\sqrt{\frac{3}{5}}}{4} \\
\{35,38\} & \frac{3\sqrt{\frac{3}{5}}}{2} \\
\{35,46\} & \frac{11\sqrt{\frac{3}{5}}}{4} \\
\{35,51\} & \frac{7\sqrt{\frac{3}{5}}}{4} \\
\{35,57\} & \frac{7\sqrt{\frac{3}{5}}}{4} \\
\{36,48\} & 1 \\
\{42,48\} & \frac{3}{2} \\
\{48,49\} & \frac{13\sqrt{\frac{7}{3}}}{8} \\
\{48,51\} & \frac{13\sqrt{\frac{7}{3}}}{8} \\
\{48,53\} & \sqrt{\frac{7}{3}} \\
\{49,63\} & \frac{3}{4}
\end{pmatrix}.
$$

V.   THE QUTRIT-QUTRIT CASE $n = 9$

Here we only have — since $9 = 3^2$ — *one* option available for computing the partial transpose, that is transposing in place the nine 3×3 blocks of the 9×9 density matrices.

We obtained the results:

$$
\begin{pmatrix}
\{1,13\} & 360 & \frac{16}{81} & \frac{\pi}{4} & 0.785398 \\[4pt]
\{3,11\} & 8 & \frac{32\sqrt{2}}{243} & \frac{1}{\sqrt{2}} & 0.707107 \\[4pt]
\{3,13\} & 8 & \frac{16}{81} & \frac{5}{6} & 0.833333 \\[4pt]
\{3,25\} & 8 & \frac{32\sqrt{2}}{243} & \frac{5}{4\sqrt{2}} & 0.883883 \\[4pt]
\{8,13\} & 8 & \frac{8}{27} & \frac{1}{\sqrt{3}} & 0.577350 \\[4pt]
\{8,25\} & 8 & \frac{4\sqrt{2}}{27} & \sqrt{\frac{2}{3}} & 0.816497 \\[4pt]
\{11,15\} & 4 & \frac{16\sqrt{2}\pi}{243} & \frac{1}{3} + \frac{3\sqrt{3}}{4\pi} & 0.746830 \\[4pt]
\{11,24\} & 2 & \frac{25\sqrt{\frac{5}{2}}}{162} & \frac{2}{5} + \frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right) & 0.863648 \\[4pt]
\{13,24\} & 2 & \frac{25\sqrt{\frac{5}{2}}}{162} & \frac{8}{75}\left(-2 + 5\sqrt{5}\right) & 0.979236 \\[4pt]
\{13,35\} & 4 & \frac{16}{15\sqrt{15}} & \frac{1}{12}\left(5 + 3\sqrt{5}\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)\right) & 0.886838 \\[4pt]
\{15,16\} & 4 & \frac{128\sqrt{2}}{729} & \frac{1}{32}\left(9\sqrt{3} + 4\pi\right) & 0.879838 \\[4pt]
\{15,44\} & 4 & \frac{32\sqrt{\frac{2}{3}}}{81} & \frac{11}{12} & 0.916667 \\[4pt]
\{15,55\} & 4 & \frac{128\sqrt{\frac{2}{3}}}{243} & \frac{11}{16} & 0.687500 \\[4pt]
\{16,24\} & 2 & \frac{25}{324}\sqrt{\frac{5}{2}}\pi & \frac{4+5\sin^{-1}\left(\frac{4}{5}\right)}{5\pi} & 0.549815 \\[4pt]
\{20,24\} & 2 & \frac{25}{324}\sqrt{\frac{5}{2}}\pi & \frac{92+75\sin^{-1}\left(\frac{4}{5}\right)}{75\pi} & 0.685627 \\[4pt]
\{24,25\} & 2 & \frac{50\sqrt{2}}{243} & 1 - \frac{2}{5\sqrt{5}} & 0.821115 \\[4pt]
\{24,27\} & 2 & \frac{50\sqrt{2}}{243} & \frac{92+75\cos^{-1}\left(\frac{3}{5}\right)}{80\sqrt{5}} & 0.903076
\end{pmatrix}
\tag{23}
$$

$$
\left(
\begin{array}{cccc}
\{24,44\} & 2 & \frac{100\sqrt{2}}{243} & \frac{1}{\sqrt{5}} & 0.447214 \\[2mm]
\{24,46\} & 2 & \frac{10\sqrt{10}}{81} & \frac{11}{15} & 0.733333 \\[2mm]
\{24,55\} & 2 & \frac{50\sqrt{2}}{243} & \frac{2}{\sqrt{5}} & 0.894427 \\[2mm]
\{24,59\} & 2 & \frac{10\sqrt{10}}{81} & \frac{14}{15} & 0.933333 \\[2mm]
\{24,70\} & 2 & \frac{50\sqrt{2}}{243} & \frac{11}{5\sqrt{5}} & 0.983870 \\[2mm]
\{24,72\} & 2 & \frac{100\sqrt{2}}{243} & \frac{7}{5\sqrt{5}} & 0.626099 \\[2mm]
\{25,35\} & 4 & \frac{4\pi}{15\sqrt{3}} & \frac{\sqrt{5}+3\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)}{3\pi} & 0.504975 \\[2mm]
\{35,46\} & 4 & \frac{32\sqrt{\frac{2}{5}}}{27} & \frac{1}{\sqrt{6}} & 0.408248 \\[2mm]
\{35,70\} & 4 & \frac{32\sqrt{2}}{135} & \sqrt{\frac{5}{6}} & 0.912871 \\[2mm]
\{38,48\} & 8 & \frac{49}{486}\sqrt{\frac{7}{2}}\pi & \frac{22\sqrt{6}+49\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)}{49\pi} & 0.596820 \\[2mm]
\{38,63\} & 8 & \frac{512}{567\sqrt{7}} & \frac{1}{16}\left(7+4\sqrt{7}\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)\right) & 0.915544 \\[2mm]
\{40,63\} & 8 & \frac{512}{567\sqrt{7}} & -\frac{49}{192}+\frac{\sqrt{14}}{3} & 0.992011 \\[2mm]
\{48,49\} & 8 & \frac{196\sqrt{2}}{729} & \frac{3\left(22\sqrt{6}+49\cos^{-1}\left(\frac{5}{7}\right)\right)}{112\sqrt{7}} & 0.930129 \\[2mm]
\{49,63\} & 8 & \frac{128\pi}{567} & \frac{\sqrt{7}+4\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)}{4\pi} & 0.440596 \\[2mm]
\{53,63\} & 8 & \frac{128\pi}{567} & \frac{19\sqrt{7}+48\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)}{48\pi} & 0.563412 \\[2mm]
\{63,64\} & 8 & \frac{512\sqrt{2}}{1701} & 1-\frac{7\sqrt{\frac{7}{2}}}{64} & 0.795378 \\[2mm]
\{63,66\} & 8 & \frac{512\sqrt{2}}{1701} & \frac{19\sqrt{7}+48\cos^{-1}\left(\frac{3}{4}\right)}{64\sqrt{2}} & 0.938690
\end{array}
\right).
$$

A. Boundary States

Here, we have for the one-dimensional sets of (exterior) boundary states the results

$$
\begin{pmatrix}
\{1,13\} & \frac{4}{9} & 0 & 0 & 0. \\
\{3,11\} & \frac{2}{3} & \frac{2}{9} & \frac{1}{3} & 0.333333 \\
\{3,13\} & \frac{4}{9} & \frac{2}{9} & \frac{1}{2} & 0.5 \\
\{3,25\} & \frac{2}{3} & \frac{2}{9} & \frac{1}{3} & 0.333333 \\
\{8,13\} & \frac{4}{3\sqrt{3}} & \frac{1}{3\sqrt{3}} & \frac{1}{4} & 0.25 \\
\{8,25\} & \frac{1}{\sqrt{3}} & \frac{2}{3\sqrt{3}} & \frac{2}{3} & 0.666667 \\
\{11,15\} & \frac{8}{9\sqrt{3}} & \frac{4}{9} & \frac{\sqrt{3}}{2} & 0.866025 \\
\{11,24\} & \frac{5}{18} & \frac{2}{9} & \frac{4}{5} & 0.8 \\
\{13,24\} & \frac{5}{9} & \frac{2\sqrt{5}}{9} & \frac{2}{\sqrt{5}} & 0.894427 \\
\{13,35\} & \frac{4}{15} & \frac{2}{9} & \frac{5}{6} & 0.833333 \\
\{15,16\} & \frac{8\sqrt{\frac{2}{3}}}{9} & \frac{2\sqrt{\frac{2}{3}}}{3} & \frac{3}{4} & 0.75 \\
\{15,44\} & \frac{8\sqrt{\frac{2}{3}}}{9} & \frac{2\sqrt{\frac{2}{3}}}{3} & \frac{3}{4} & 0.75 \\
\{15,55\} & \frac{10\sqrt{\frac{2}{3}}}{9} & \frac{2\sqrt{\frac{2}{3}}}{9} & \frac{1}{5} & 0.2 \\
\{16,24\} & \frac{5}{9} & \frac{4}{9} & \frac{4}{5} & 0.8 \\
\{20,24\} & \frac{5}{9} & \frac{4}{9} & \frac{4}{5} & 0.8 \\
\{24,25\} & \frac{5\sqrt{\frac{5}{2}}}{9} & \frac{\sqrt{\frac{5}{2}}}{9} & \frac{1}{5} & 0.2 \\
\{24,27\} & \frac{5\sqrt{\frac{5}{2}}}{9} & \frac{2\sqrt{10}}{9} & \frac{4}{5} & 0.8
\end{pmatrix}
\tag{24}
$$

$$
\begin{pmatrix}
\{24,44\} & \frac{\sqrt{10}}{3} & \frac{\sqrt{\frac{5}{2}}}{9} & \frac{1}{6} & 0.166667 \\
\{24,46\} & \frac{5\sqrt{\frac{5}{2}}}{9} & \frac{\sqrt{\frac{5}{2}}}{9} & \frac{1}{5} & 0.2 \\
\{24,55\} & \frac{5\sqrt{\frac{5}{2}}}{9} & \frac{2\sqrt{10}}{9} & \frac{4}{5} & 0.8 \\
\{24,59\} & \frac{5\sqrt{\frac{5}{2}}}{9} & \frac{2\sqrt{10}}{9} & \frac{4}{5} & 0.8 \\
\{24,70\} & \frac{5\sqrt{\frac{5}{2}}}{9} & \frac{2\sqrt{10}}{9} & \frac{4}{5} & 0.8 \\
\{24,72\} & \frac{\sqrt{10}}{3} & \frac{\sqrt{\frac{5}{2}}}{9} & \frac{1}{6} & 0.166667 \\
\{25,35\} & \frac{4}{3\sqrt{5}} & \frac{4}{9} & \frac{\sqrt{5}}{3} & 0.745356 \\
\{35,46\} & \frac{14}{3\sqrt{15}} & \frac{2}{3\sqrt{15}} & \frac{1}{7} & 0.142857 \\
\{35,70\} & \frac{4}{\sqrt{15}} & \frac{2\sqrt{\frac{5}{3}}}{3} & \frac{5}{6} & 0.833333 \\
\{38,48\} & \frac{7\sqrt{\frac{2}{3}}}{9} & \frac{4}{9} & \frac{2\sqrt{6}}{7} & 0.699854 \\
\{38,63\} & \frac{16}{63} & \frac{2}{9} & \frac{7}{8} & 0.875 \\
\{40,63\} & \frac{32}{63} & \frac{8\sqrt{\frac{2}{7}}}{9} & \frac{\sqrt{\frac{7}{2}}}{2} & 0.935414 \\
\{48,49\} & \frac{7\sqrt{\frac{7}{3}}}{9} & \frac{2\sqrt{\frac{7}{3}}}{3} & \frac{6}{7} & 0.857143 \\
\{49,63\} & \frac{16}{9\sqrt{7}} & \frac{4}{9} & \frac{\sqrt{7}}{4} & 0.661438 \\
\{53,63\} & \frac{16}{9\sqrt{7}} & \frac{4}{9} & \frac{\sqrt{7}}{4} & 0.661438 \\
\{63,64\} & \frac{32}{9\sqrt{7}} & \frac{4}{9\sqrt{7}} & \frac{1}{8} & 0.125 \\
\{63,66\} & \frac{32}{9\sqrt{7}} & \frac{4\sqrt{7}}{9} & \frac{7}{8} & 0.875
\end{pmatrix}.
$$

B. Length of Separability-Nonseparability *Interior* Boundary

In the following arrays, we present the HS-length (and now a numerical approximation to it) of the common border separating the states lacking a PPR from those that possess a

PPT for each specific scenario.

$$
\begin{pmatrix}
\{1,13\} & \frac{4}{9} & 0.444444 \\
\{3,11\} & \frac{4}{9\sqrt{3}} & 0.2566 \\
\{3,13\} & \frac{4}{9} & 0.444444 \\
\{3,25\} & \frac{4}{9\sqrt{3}} & 0.2566 \\
\{8,13\} & \frac{2}{27}\left(-2\sqrt{3}+3\sqrt{6}\right) & 0.287731 \\
\{8,25\} & \frac{1}{297}\left(11\sqrt{3}+12\sqrt{33}\right) & 0.296254 \\
\{11,15\} & \frac{4}{9} & 0.444444 \\
\{11,24\} & \frac{4}{11} & 0.363636 \\
\{13,24\} & \frac{4}{11} & 0.363636 \\
\{13,35\} & \frac{7}{18} & 0.388889 \\
\{15,16\} & \frac{1}{621}\left(23\sqrt{6}+12\sqrt{138}\right) & 0.317724 \\
\{15,44\} & \frac{2}{27}\left(3+\sqrt{6}\right) & 0.403666 \\
\{15,55\} & \frac{1}{45}\left(-5\sqrt{6}+8\sqrt{10}\right) & 0.290017 \\
\{16,24\} & \frac{4}{9} & 0.444444 \\
\{20,24\} & \frac{4}{9} & 0.444444 \\
\{24,25\} & \frac{117\sqrt{10}+40\sqrt{390}}{3510} & 0.330462 \\
\{24,27\} & \frac{117\sqrt{10}+40\sqrt{390}}{3510} & 0.330462
\end{pmatrix}
\tag{25}
$$

$$
\begin{pmatrix}
\{24,44\} & -\frac{4}{135}\left(3\sqrt{10}-5\sqrt{15}\right) & 0.292684 \\
\{24,46\} & \frac{1}{18}\left(4+\sqrt{10}\right) & 0.397904 \\
\{24,55\} & \frac{117\sqrt{10}+40\sqrt{390}}{3510} & 0.330462 \\
\{24,59\} & \frac{1}{18}\left(4+\sqrt{10}\right) & 0.397904 \\
\{24,70\} & \frac{117\sqrt{10}+40\sqrt{390}}{3510} & 0.330462 \\
\{24,72\} & -\frac{4}{135}\left(3\sqrt{10}-5\sqrt{15}\right) & 0.292684 \\
\{25,35\} & \frac{4}{9} & 0.444444 \\
\{35,46\} & -\frac{2}{189}\left(7\sqrt{15}-12\sqrt{21}\right) & 0.295027 \\
\{35,70\} & \frac{4\left(59\sqrt{15}+15\sqrt{885}\right)}{7965} & 0.338853 \\
\{38,48\} & \frac{4}{9} & 0.444444 \\
\{38,63\} & \frac{12}{29} & 0.413793 \\
\{40,63\} & \frac{12}{29} & 0.413793 \\
\{48,49\} & \frac{415\sqrt{21}+84\sqrt{1743}}{15687} & 0.344789 \\
\{49,63\} & \frac{4}{9} & 0.444444 \\
\{53,63\} & \frac{4}{9} & 0.444444 \\
\{63,64\} & \frac{333\sqrt{7}+56\sqrt{777}}{6993} & 0.349209 \\
\{63,66\} & \frac{333\sqrt{7}+56\sqrt{777}}{6993} & 0.349209 \\
\end{pmatrix}.
$$

VI.   THE CASE $n=10$

A. 5×2 Decomposition

We first compute the partial transpose by transposing in place the (2^2) four 5×5 blocks of our set of two-dimensional 10×10 density matrices. Our analysis yielded for the HS-total

volumes and PPT-probabilities

$$
\begin{pmatrix}
\{1,29\} & 480 & \frac{4}{25} & \frac{\pi}{4} & 0.785398 \\[4pt]
\{3,27\} & 4 & \frac{8\sqrt{2}}{75} & \frac{1}{\sqrt{2}} & 0.707107 \\[4pt]
\{3,29\} & 12 & \frac{4}{25} & \frac{5}{6} & 0.833333 \\[4pt]
\{3,49\} & 12 & \frac{8\sqrt{2}}{75} & \frac{5}{4\sqrt{2}} & 0.883883 \\[4pt]
\{8,29\} & 4 & \frac{6}{25} & \frac{1}{\sqrt{3}} & 0.577350 \\[4pt]
\{8,31\} & 8 & \frac{3\sqrt{3}}{25} & \frac{7}{9} & 0.777778 \\[4pt]
\{8,49\} & 4 & \frac{3\sqrt{2}}{25} & \sqrt{\frac{2}{3}} & 0.816497 \\[4pt]
\{8,55\} & 4 & \frac{3\sqrt{3}}{25} & \frac{8}{9} & 0.888889 \\[4pt]
\{8,64\} & 8 & \frac{3\sqrt{2}}{25} & \frac{7}{3\sqrt{6}} & 0.952579 \\[4pt]
\{8,68\} & 4 & \frac{6}{25} & \frac{4}{3\sqrt{3}} & 0.769800 \\[4pt]
\{15,31\} & 6 & \frac{32\sqrt{\frac{2}{3}}}{75} & \frac{1}{2} & 0.500000 \\[4pt]
\{15,33\} & 6 & \frac{8\sqrt{\frac{2}{3}}}{25} & \frac{3}{4} & 0.750000 \\[4pt]
\{15,64\} & 6 & \frac{32\sqrt{2}}{225} & \frac{\sqrt{3}}{2} & 0.866025 \\[4pt]
\{15,72\} & 2 & \frac{8\sqrt{\frac{2}{3}}}{25} & \frac{11}{12} & 0.916667 \\[4pt]
\{15,81\} & 6 & \frac{32\sqrt{2}}{225} & \frac{9\sqrt{3}}{16} & 0.974279 \\[4pt]
\{15,87\} & 2 & \frac{32\sqrt{\frac{2}{3}}}{75} & \frac{11}{16} & 0.687500 \\[4pt]
\{24,33\} & 8 & \frac{\sqrt{2}}{3} & \frac{1}{\sqrt{5}} & 0.447214 \\[4pt]
\{24,81\} & 8 & \frac{1}{3\sqrt{2}} & \frac{2}{\sqrt{5}} & 0.894427 \\[4pt]
\{27,35\} & 8 & \frac{9\sqrt{3}\pi}{125} & \frac{14\sqrt{5}+27\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)}{27\pi} & 0.636783 \\[4pt]
\{27,48\} & 2 & \frac{49\sqrt{\frac{7}{3}}}{300} & \frac{1}{28}\left(12+7\sqrt{6}\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)\right) & 0.903278
\end{pmatrix}
\tag{26}
$$

$$
\begin{pmatrix}
\{29,48\} & 6 & \frac{49\sqrt{\frac{7}{3}}}{300} & \frac{4}{147}\left(-9+7\sqrt{42}\right) & 0.989529 \\[2mm]
\{29,63\} & 4 & \frac{128}{175\sqrt{7}} & \frac{1}{16}\left(7+4\sqrt{7}\,\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)\right) & 0.915544 \\[2mm]
\{31,63\} & 8 & \frac{128}{175\sqrt{7}} & -\frac{49}{192}+\frac{\sqrt{14}}{3} & 0.992011 \\[2mm]
\{31,80\} & 6 & \frac{243}{800} & \frac{4}{9}+\frac{\sin^{-1}\left(\frac{4\sqrt{2}}{9}\right)}{\sqrt{2}} & 0.925046 \\[2mm]
\{33,80\} & 6 & \frac{243}{800} & \frac{8}{243}\left(-8+27\sqrt{2}\right) & 0.993704 \\[2mm]
\{33,99\} & 8 & \frac{4\sqrt{5}}{27} & \frac{3}{20}\left(3+5\sin^{-1}\left(\frac{3}{5}\right)\right) & 0.932626 \\[2mm]
\{35,36\} & 8 & \frac{24\sqrt{2}}{125} & \frac{14\sqrt{5}+27\cos^{-1}\left(\frac{2}{3}\right)}{24\sqrt{6}} & 0.918793 \\[2mm]
\{36,48\} & 2 & \frac{49}{600}\sqrt{\frac{7}{2}}\,\pi & \frac{2\sqrt{6}+7\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)}{7\pi} & 0.469522 \\[2mm]
\{40,48\} & 6 & \frac{49}{600}\sqrt{\frac{7}{2}}\,\pi & \frac{22\sqrt{6}+49\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)}{49\pi} & 0.596820 \\[2mm]
\{48,49\} & 6 & \frac{49\sqrt{2}}{225} & 1-\frac{3\sqrt{\frac{3}{14}}}{7} & 0.801610 \\[2mm]
\{48,51\} & 6 & \frac{49\sqrt{2}}{225} & \frac{3\left(22\sqrt{6}+49\cos^{-1}\left(\frac{5}{7}\right)\right)}{112\sqrt{7}} & 0.930129 \\[2mm]
\{49,63\} & 4 & \frac{32\pi}{175} & \frac{\sqrt{7}+4\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)}{4\pi} & 0.440596 \\[2mm]
\{55,63\} & 4 & \frac{32\pi}{175} & \frac{19\sqrt{7}+48\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)}{48\pi} & 0.563412 \\[2mm]
\{63,64\} & 8 & \frac{128\sqrt{2}}{525} & 1-\frac{7\sqrt{\frac{7}{2}}}{64} & 0.795378 \\[2mm]
\{63,68\} & 4 & \frac{128\sqrt{2}}{525} & \frac{19\sqrt{7}+48\cos^{-1}\left(\frac{3}{4}\right)}{64\sqrt{2}} & 0.938690 \\[2mm]
\{64,80\} & 6 & \frac{243\pi}{800\sqrt{2}} & \frac{4\sqrt{2}+9\sin^{-1}\left(\frac{4\sqrt{2}}{9}\right)}{9\pi} & 0.416417 \\[2mm]
\{72,80\} & 2 & \frac{243\pi}{800\sqrt{2}} & \frac{\frac{172\sqrt{2}}{243}+\sin^{-1}\left(\frac{4\sqrt{2}}{9}\right)}{\pi} & 0.534977 \\[2mm]
\{80,81\} & 6 & \frac{27}{50\sqrt{2}} & 1-\frac{4\sqrt{2}}{27} & 0.790487 \\[2mm]
\{80,87\} & 2 & \frac{27}{50\sqrt{2}} & \frac{43}{54\sqrt{2}}+\frac{9}{16}\cos^{-1}\left(\frac{7}{9}\right) & 0.945383 \\[2mm]
\{81,99\} & 8 & \frac{\sqrt{5}\pi}{9} & \frac{3+5\sin^{-1}\left(\frac{3}{5}\right)}{5\pi} & 0.395819
\end{pmatrix}.
$$

B. 2×5 Decomposition

Now, we compute the partial transpose by transposing in place the (5^2) twenty-five 4×4 blocks of the 10×10 density matrices.

$$
\begin{pmatrix}
\{3,6\} & 16 & \frac{8\sqrt{2}}{75} & \frac{1}{\sqrt{2}} & 0.707107 \\
\{4,18\} & 480 & \frac{4}{25} & \frac{\pi}{4} & 0.785398 \\
\{6,8\} & 2 & \frac{9}{200}\sqrt{\frac{3}{2}}\pi & \frac{26\sqrt{2}+27\tan^{-1}\left(2\sqrt{2}\right)}{27\pi} & 0.825312 \\
\{6,15\} & 2 & \frac{16\sqrt{\frac{2}{3}}}{75} & \frac{1}{24}\left(9+2\sqrt{3}\pi\right) & 0.828450 \\
\{8,9\} & 2 & \frac{3\sqrt{2}}{25} & \frac{52+27\sqrt{2}\sec^{-1}(3)}{48\sqrt{6}} & 0.842035 \\
\{8,22\} & 6 & \frac{3\sqrt{3}}{25} & \frac{8}{9} & 0.888889 \\
\{8,29\} & 6 & \frac{6}{25} & \frac{4}{3\sqrt{3}} & 0.769800 \\
\{9,15\} & 2 & \frac{4\sqrt{2}\pi}{75} & \frac{1}{3}+\frac{\sqrt{3}}{2\pi} & 0.608998 \\
\{15,22\} & 6 & \frac{32\sqrt{\frac{2}{3}}}{75} & \frac{1}{2} & 0.500000 \\
\{15,29\} & 6 & \frac{32\sqrt{2}}{225} & \frac{\sqrt{3}}{2} & 0.866025 \\
\{18,24\} & 4 & \frac{1}{16}\sqrt{\frac{5}{2}}\pi & \frac{92+75\sin^{-1}\left(\frac{4}{5}\right)}{75\pi} & 0.685627 \\
\{18,35\} & 4 & \frac{36\sqrt{\frac{3}{5}}}{125} & \frac{1}{12}\left(5+3\sqrt{5}\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)\right) & 0.886838 \\
\{24,25\} & 4 & \frac{1}{3\sqrt{2}} & \frac{92+75\cos^{-1}\left(\frac{3}{5}\right)}{80\sqrt{5}} & 0.903076 \\
\{24,46\} & 4 & \frac{1}{\sqrt{10}} & \frac{14}{15} & 0.933333 \\
\{24,57\} & 4 & \frac{\sqrt{2}}{3} & \frac{7}{5\sqrt{5}} & 0.626099
\end{pmatrix}
\tag{27}
$$

$$\begin{pmatrix}
\{25,35\} & 4 & \frac{9\sqrt{3}\pi}{125} & \frac{\sqrt{5}+3\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)}{3\pi} & 0.504975 \\[2ex]
\{35,46\} & 4 & \frac{24\sqrt{\frac{2}{5}}}{25} & \frac{1}{\sqrt{6}} & 0.408248 \\[2ex]
\{35,57\} & 4 & \frac{24\sqrt{2}}{125} & \sqrt{\frac{5}{6}} & 0.912871 \\[2ex]
\{38,48\} & 6 & \frac{49}{600}\sqrt{\frac{7}{2}}\pi & \frac{22\sqrt{6}+49\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)}{49\pi} & 0.596820 \\[2ex]
\{38,63\} & 6 & \frac{128}{175\sqrt{7}} & \frac{1}{16}\left(7+4\sqrt{7}\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)\right) & 0.915544 \\[2ex]
\{48,49\} & 6 & \frac{49\sqrt{2}}{225} & \frac{3\left(22\sqrt{6}+49\cos^{-1}\left(\frac{5}{7}\right)\right)}{112\sqrt{7}} & 0.930129 \\[2ex]
\{48,78\} & 2 & \frac{7\sqrt{\frac{7}{3}}}{25} & \frac{20}{21} & 0.952381 \\[2ex]
\{48,93\} & 2 & \frac{98}{75\sqrt{3}} & \frac{10}{7\sqrt{7}} & 0.539949 \\[2ex]
\{49,63\} & 6 & \frac{32\pi}{175} & \frac{\sqrt{7}+4\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)}{4\pi} & 0.440596 \\[2ex]
\{63,78\} & 2 & \frac{128\sqrt{\frac{2}{7}}}{75} & \frac{1}{2\sqrt{2}} & 0.353553 \\[2ex]
\{63,93\} & 2 & \frac{128\sqrt{2}}{525} & \frac{\sqrt{\frac{7}{2}}}{2} & 0.935414 \\[2ex]
\{66,80\} & 8 & \frac{243\pi}{800\sqrt{2}} & \frac{\frac{172\sqrt{2}}{243}+\sin^{-1}\left(\frac{4\sqrt{2}}{9}\right)}{\pi} & 0.534977 \\[2ex]
\{66,99\} & 8 & \frac{4\sqrt{5}}{27} & \frac{3}{20}\left(3+5\sin^{-1}\left(\frac{3}{5}\right)\right) & 0.932626 \\[2ex]
\{80,81\} & 8 & \frac{27}{50\sqrt{2}} & \frac{43}{54\sqrt{2}}+\frac{9}{16}\cos^{-1}\left(\frac{7}{9}\right) & 0.945383 \\[2ex]
\{81,99\} & 8 & \frac{\sqrt{5}\pi}{9} & \frac{3+5\sin^{-1}\left(\frac{3}{5}\right)}{5\pi} & 0.395819
\end{pmatrix}.$$

C. Bi-PPT

Now, we require that the 10×10 density matrices be positive under *both* the forms of partial transposition employed immediately above. We obtained the (extensive) results

$$
\begin{pmatrix}
\{1,29\} & 904 & \frac{4}{25} & \frac{\pi}{4} & 0.785398 \\[4pt]
\{3,6\} & 20 & \frac{8\sqrt{2}}{75} & \frac{1}{\sqrt{2}} & 0.707107 \\[4pt]
\{3,29\} & 12 & \frac{4}{25} & \frac{5}{6} & 0.833333 \\[4pt]
\{3,51\} & 6 & \frac{8\sqrt{2}}{75} & \frac{5}{4\sqrt{2}} & 0.883883 \\[4pt]
\{6,8\} & 2 & \frac{9}{200}\sqrt{\frac{3}{2}}\pi & \frac{26\sqrt{2}+27\tan^{-1}\left(2\sqrt{2}\right)}{27\pi} & 0.825312 \\[4pt]
\{6,15\} & 2 & \frac{16\sqrt{\frac{2}{3}}}{75} & \frac{1}{24}\left(9+2\sqrt{3}\pi\right) & 0.828450 \\[4pt]
\{8,9\} & 2 & \frac{3\sqrt{2}}{25} & \frac{52+27\sqrt{2}\sec^{-1}(3)}{48\sqrt{6}} & 0.842035 \\[4pt]
\{8,22\} & 8 & \frac{3\sqrt{3}}{25} & \frac{8}{9} & 0.888889 \\[4pt]
\{8,29\} & 4 & \frac{6}{25} & \frac{1}{\sqrt{3}} & 0.577350 \\[4pt]
\{8,31\} & 6 & \frac{3\sqrt{3}}{25} & \frac{7}{9} & 0.777778 \\[4pt]
\{8,42\} & 2 & \frac{3\sqrt{3}}{25} & \frac{2}{3} & 0.666667 \\[4pt]
\{8,49\} & 4 & \frac{3\sqrt{2}}{25} & \sqrt{\frac{2}{3}} & 0.816497 \\[4pt]
\{8,53\} & 6 & \frac{6}{25} & \frac{4}{3\sqrt{3}} & 0.769800 \\[4pt]
\{8,64\} & 8 & \frac{3\sqrt{2}}{25} & \frac{7}{3\sqrt{6}} & 0.952579 \\[4pt]
\{9,15\} & 2 & \frac{4\sqrt{2}\pi}{75} & \frac{1}{3}+\frac{\sqrt{3}}{2\pi} & 0.608998 \\[4pt]
\{15,22\} & 10 & \frac{32\sqrt{\frac{2}{3}}}{75} & \frac{1}{2} & 0.500000 \\[4pt]
\{15,29\} & 12 & \frac{32\sqrt{2}}{225} & \frac{\sqrt{3}}{2} & 0.866025 \\[4pt]
\{15,33\} & 6 & \frac{8\sqrt{\frac{2}{3}}}{25} & \frac{3}{4} & 0.750000 \\[4pt]
\{15,72\} & 2 & \frac{8\sqrt{\frac{2}{3}}}{25} & \frac{11}{12} & 0.916667 \\[4pt]
\{15,81\} & 4 & \frac{32\sqrt{2}}{225} & \frac{9\sqrt{3}}{16} & 0.974279
\end{pmatrix}
\tag{28}
$$

$$
\begin{pmatrix}
\{15,87\} & 2 & \frac{32\sqrt{\frac{2}{3}}}{75} & \frac{11}{16} & 0.687500 \\[2mm]
\{18,24\} & 4 & \frac{1}{16}\sqrt{\frac{5}{2}}\pi & \frac{92+75\sin^{-1}\left(\frac{4}{5}\right)}{75\pi} & 0.685627 \\[2mm]
\{18,35\} & 4 & \frac{36\sqrt{\frac{3}{5}}}{125} & \frac{1}{12}\left(5+3\sqrt{5}\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)\right) & 0.886838 \\[2mm]
\{24,25\} & 4 & \frac{1}{3\sqrt{2}} & \frac{92+75\cos^{-1}\left(\frac{3}{5}\right)}{80\sqrt{5}} & 0.903076 \\[2mm]
\{24,33\} & 8 & \frac{\sqrt{2}}{3} & \frac{1}{\sqrt{5}} & 0.447214 \\[2mm]
\{24,46\} & 4 & \frac{1}{\sqrt{10}} & \frac{14}{15} & 0.933333 \\[2mm]
\{24,81\} & 8 & \frac{1}{3\sqrt{2}} & \frac{2}{\sqrt{5}} & 0.894427 \\[2mm]
\{24,89\} & 2 & \frac{\sqrt{2}}{3} & \frac{7}{5\sqrt{5}} & 0.626099 \\[2mm]
\{25,35\} & 4 & \frac{9\sqrt{3}\pi}{125} & \frac{\sqrt{5}+3\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)}{3\pi} & 0.504975 \\[2mm]
\{27,35\} & 6 & \frac{9\sqrt{3}\pi}{125} & \frac{14\sqrt{5}+27\csc^{-1}\left(\frac{3}{\sqrt{5}}\right)}{27\pi} & 0.636783 \\[2mm]
\{27,48\} & 2 & \frac{49\sqrt{\frac{7}{3}}}{300} & \frac{1}{28}\left(12+7\sqrt{6}\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)\right) & 0.903278 \\[2mm]
\{29,48\} & 6 & \frac{49\sqrt{\frac{7}{3}}}{300} & \frac{4}{147}\left(-9+7\sqrt{42}\right) & 0.989529 \\[2mm]
\{29,63\} & 10 & \frac{128}{175\sqrt{7}} & \frac{1}{16}\left(7+4\sqrt{7}\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)\right) & 0.915544 \\[2mm]
\{31,63\} & 6 & \frac{128}{175\sqrt{7}} & -\frac{49}{192}+\frac{\sqrt{14}}{3} & 0.992011 \\[2mm]
\{31,80\} & 6 & \frac{243}{800} & \frac{4}{9}+\frac{\sin^{-1}\left(\frac{4\sqrt{2}}{9}\right)}{\sqrt{2}} & 0.925046 \\[2mm]
\{33,80\} & 6 & \frac{243}{800} & \frac{8}{243}\left(-8+27\sqrt{2}\right) & 0.993704 \\[2mm]
\{33,99\} & 16 & \frac{4\sqrt{5}}{27} & \frac{3}{20}\left(3+5\sin^{-1}\left(\frac{3}{5}\right)\right) & 0.932626 \\[2mm]
\{35,36\} & 8 & \frac{24\sqrt{2}}{125} & \frac{14\sqrt{5}+27\cos^{-1}\left(\frac{2}{3}\right)}{24\sqrt{6}} & 0.918793 \\[2mm]
\{35,46\} & 4 & \frac{24\sqrt{\frac{2}{5}}}{25} & \frac{1}{\sqrt{6}} & 0.408248 \\[2mm]
\{35,57\} & 4 & \frac{24\sqrt{2}}{125} & \sqrt{\frac{5}{6}} & 0.912871
\end{pmatrix}
$$

$$\begin{pmatrix}
\{36,48\} & 2 & \frac{49}{600}\sqrt{\frac{7}{2}}\pi & \frac{2\sqrt{6}+7\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)}{7\pi} & 0.469522 \\[2mm]
\{38,48\} & 10 & \frac{49}{600}\sqrt{\frac{7}{2}}\pi & \frac{22\sqrt{6}+49\sin^{-1}\left(\frac{2\sqrt{6}}{7}\right)}{49\pi} & 0.596820 \\[2mm]
\{48,49\} & 2 & \frac{49\sqrt{2}}{225} & \frac{3\left(2\sqrt{6}+7\cos^{-1}\left(\frac{5}{7}\right)\right)}{16\sqrt{7}} & 0.731739 \\[2mm]
\{48,51\} & 10 & \frac{49\sqrt{2}}{225} & \frac{3\left(22\sqrt{6}+49\cos^{-1}\left(\frac{5}{7}\right)\right)}{112\sqrt{7}} & 0.930129 \\[2mm]
\{48,64\} & 4 & \frac{49\sqrt{2}}{225} & 1-\frac{3\sqrt{\frac{3}{14}}}{7} & 0.801610 \\[2mm]
\{48,78\} & 2 & \frac{7\sqrt{\frac{7}{3}}}{25} & \frac{20}{21} & 0.952381 \\[2mm]
\{48,93\} & 2 & \frac{98}{75\sqrt{3}} & \frac{10}{7\sqrt{7}} & 0.539949 \\[2mm]
\{49,63\} & 8 & \frac{32\pi}{175} & \frac{\sqrt{7}+4\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)}{4\pi} & 0.440596 \\[2mm]
\{55,63\} & 2 & \frac{32\pi}{175} & \frac{19\sqrt{7}+48\csc^{-1}\left(\frac{4}{\sqrt{7}}\right)}{48\pi} & 0.563412 \\[2mm]
\{63,64\} & 8 & \frac{128\sqrt{2}}{525} & 1-\frac{7\sqrt{\frac{7}{2}}}{64} & 0.795378 \\[2mm]
\{63,68\} & 4 & \frac{128\sqrt{2}}{525} & \frac{19\sqrt{7}+48\cos^{-1}\left(\frac{3}{4}\right)}{64\sqrt{2}} & 0.938690 \\[2mm]
\{63,78\} & 2 & \frac{128\sqrt{\frac{2}{7}}}{75} & \frac{1}{2\sqrt{2}} & 0.353553 \\[2mm]
\{63,93\} & 2 & \frac{128\sqrt{2}}{525} & \frac{\sqrt{\frac{7}{2}}}{2} & 0.935414 \\[2mm]
\{64,80\} & 6 & \frac{243\pi}{800\sqrt{2}} & \frac{4\sqrt{2}+9\sin^{-1}\left(\frac{4\sqrt{2}}{9}\right)}{9\pi} & 0.416417 \\[2mm]
\{70,80\} & 8 & \frac{243\pi}{800\sqrt{2}} & \frac{\frac{172\sqrt{2}}{243}+\sin^{-1}\left(\frac{4\sqrt{2}}{9}\right)}{\pi} & 0.534977 \\[2mm]
\{80,81\} & 4 & \frac{27}{50\sqrt{2}} & \frac{1}{16}\left(4\sqrt{2}+9\cos^{-1}\left(\frac{7}{9}\right)\right) & 0.735870 \\[2mm]
\{80,83\} & 2 & \frac{27}{50\sqrt{2}} & 1-\frac{4\sqrt{2}}{27} & 0.790487 \\[2mm]
\{80,87\} & 6 & \frac{27}{50\sqrt{2}} & \frac{43}{54\sqrt{2}}+\frac{9}{16}\cos^{-1}\left(\frac{7}{9}\right) & 0.945383 \\[2mm]
\{81,99\} & 12 & \frac{\sqrt{5}\pi}{9} & \frac{3+5\sin^{-1}\left(\frac{3}{5}\right)}{5\pi} & 0.395819
\end{pmatrix}$$

The probability 0.993704 (corresponding to the pair of Lie generators numbered $\{33,80\}$) is the largest of any recorded in all our results above. (This also occurs in (26).)

VII. ANALYSES OF SCENARIOS WITH *MORE* THAN TWO PARAMETERS

We found it considerably simpler to extend the Jakóbczyk-Siennicki model [4] from two-qubit systems ($n = 4$) to higher n — as illustrated above — than to extend it from two-dimensional sections of Bloch vectors to m-dimensional sections ($m \geq 3$), even just for the case $n = 4$. (However, we were able to obtain a highly interesting set of *exact* HS separability probabilites for certain $m = 3, n = 4$ systems, using the Jaynes maximum-entropy principle in conjunction with the new integration over implicitly defined regions feature of Mathe-

matica, in [9, Fig. 11].) *A fortiori*, it appears that the determination of the HS separable volume of the *fifteen*-dimensional convex set of 4×4 density matrices — conjectured on the basis of an extensive quasi-Monte Carlo analysis in to be $(3^3 5^7 \sqrt{3})^{-1} \approx 2.73707 \cdot 10^{-7}$ [6, eq. (41)] — would have to proceed in some quite different analytic fashion to that pursued here. (Based on our experience in the above-reported analyses, it appears to be a necessary condition for obtaining exact HS separability/PPT-probabilities that explicit formulas be available for the eigenvectors of both the class of density matrices under consideration *and* of their partial transposes.)

A. $m = 3, n = 4$

We have been able to find, up to this point in time, that for the *three*-dimensional two-qubit ($m = 3, n = 4$) scenarios generated by the four *triads* of Gell-Mann matrices $\{1, 4, 6\}$, $\{1, 5, 7\}$ $\{2, 4, 7\}$ and $\{2, 5, 6\}$, the volume of separable states is — having to resort to numerical methods — 0.478512 and of all the (separable and nonseparable/entangled) states, 0.61685, for a separability probability of 0.775734. For the scenario $\{10, 12, 13\}$, the separable volume remains the same, but the total volume is *exactly* $\frac{\pi}{6} \approx 0.523599$ for an HS separability probability of 0.913891.

B. Two-Dimensional Boundaries of $m = 3, n = 4$ Systems

Of course, it we restrict attention to the generic boundary states of the three-dimensional scenarios, we only have to perform two-dimensional computations. Thus, we were able to find that for the *triadic* scenarios $\{1, 3, 6\}, \{1, 3, 7\}, \{1, 3, 9\}, \{1, 3, 10\}, \{1, 4, 9\}$ and $\{1, 5, 10\}$, amongst others, the HS-area of the states with degenerate spectra is $\frac{\pi}{8}$ and that of the separable component of this area, one-half that value. For the triadic scenarios $\{1, 4, 6\}$ and $\{1, 5, 7\}$, the separable component of the boundary states has area $\frac{\pi}{4}$ and the total area is $\frac{1}{2}\left(\sqrt{5} + 6 \sin^{-1}(\frac{1}{\sqrt{6}})\right)$ for a separability probability of 0.165025. Also, for several scenarios (for instance, $\{3, 4, 9\}$), we have a total area of $\frac{2\sqrt{2}}{3}$, a separable area of $\frac{1}{3}$, giving a separability probability of $\frac{1}{2\sqrt{2}} \approx 0.353553$.

Now, we present all our results of this type (two-dimensional exterior boundaries of three-dimensional scenarios) in the following array. The first column gives the corresponding

triad of Gell-Mann matrices, the second column shows the total HS-area of the boundary states, the third column gives the exact separability probability, and the last, a numerical approximation to the probability. (There may exist additional nontrivial scenarios, as we were not readily able to fully analyze all $2730 = 13 \cdot 14 \cdot 15$ possible triads of 4×4 Gell-Mann matrices.)

$$
\begin{pmatrix}
\{1,3,6\} & \frac{\pi}{4} & \frac{1}{2} & 0.500000 \\
\{1,4,6\} & \frac{1}{2}\left(\sqrt{5}+6\sin^{-1}\left(\frac{1}{\sqrt{6}}\right)\right) & \frac{\pi}{4\sqrt{5}+24\csc^{-1}\left(\sqrt{6}\right)} & 0.165025 \\
\{3,4,6\} & \frac{1}{2}\left(\sqrt{5}+6\sin^{-1}\left(\frac{1}{\sqrt{6}}\right)\right) & \frac{1}{2}+\frac{4\left(-1+\sqrt{2}\right)}{3\left(\sqrt{5}+6\csc^{-1}\left(\sqrt{6}\right)\right)} & 0.616044 \\
\{3,4,9\} & \frac{2\sqrt{2}}{3} & \frac{1}{2\sqrt{2}} & 0.353553 \\
\{3,6,7\} & \frac{1}{2}\left(\sqrt{5}+6\sin^{-1}\left(\frac{1}{\sqrt{6}}\right)\right) & \frac{1}{2} & 0.500000 \\
\{3,6,8\} & \frac{3\pi}{2} & -\frac{\sqrt{15}-8\pi+8\tan^{-1}\left(\sqrt{\frac{3}{5}}\right)}{8\pi} & 0.636114 \\
\{6,9,15\} & \frac{3\pi}{4} & \frac{4-2\sqrt{5}+3\pi-12\csc^{-1}\left(\sqrt{6}\right)}{6\pi} & 0.207232 \\
\{8,9,10\} & \frac{3}{16}\left(4+\sqrt{7}+2\pi+8\cot^{-1}\left(\sqrt{7}\right)\right) & \frac{2(2+\pi)}{4+\sqrt{7}+2\pi+8\cot^{-1}\left(\sqrt{7}\right)} & 0.650017 \\
\{9,11,13\} & \frac{3\pi}{2} & \frac{1}{12} & 0.0833333
\end{pmatrix}. \quad (29)
$$

C. 9- and 15-Parameter Analyses ($m = 9, 15, n = 4$)

Now, we sought to make some progress in obtaining the (conjecturally exact) Hilbert-Schmidt volume of the separable 4×4 density matrices, in both the 9-dimensional case of real density matrices and the 15-dimensional case of (fully general) complex density matrices. In both cases, we dispensed with the Bloch vector parameterization [1, 2, 3] used in the above analyses (neither did we employ the integration over implicitly defined regions capabilities of Mathematica version 5.1), and adopted a simple, naive parameterization, in which the four diagonal elements of the density matrices were denoted $a, b, c, 1 - a - b - c$ and the off-diagonal (upper triangular) elements, $\alpha_{ij} + i\beta_{ij}$ (in the real $m = 9$ case, of course, all β's equal zero). (In order to compare our results here with the HS-volume formulas of Życzkowski and Sommers [5], the volumes we do report below are our initial volumes multiplied by factors of 2^7 in the complex case, and 2^4 in the real case. In all our earlier analyses above, we have simply taken the HS-volume element to equal 1.)

In both ($m = 9, 15$) of these cases, we pursued the same analytical strategy. We required that the six principal 2×2 minors of the density matrices and/or their partial transposes

have nonnegative determinants. This is (only) one of the requirements for nonnegative-definiteness (cf. [12, eq. (12)]). Ideally, we would also have required that the leading principal 3×3 minor have nonnegative determinant and also that the determinant of the matrix be nonnegative. But these last two requirements were too computationally onerous to impose (at least in our first round of efforts). So, our analytical strategy should yield *upper* bounds on the Hilbert-Schmidt volumes in these cases.

1. 9-Dimensional Real Case

When we only required that the six principal 2×2 minors have nonnegative determinants, we obtained for the volume the result $\frac{\pi^2}{1120} \approx 0.00881215$. (We can reduce [improve] this to $\frac{\pi^2(16+\pi^2)}{35840} \approx 0.00712396$ by modifying [narrowing], to begin with, the integration limits over a *single* off-diagonal variable, so that in addition, to its corresponding 2×2 minor, a corresponding 3×3 minor also has a nonnegative determinant. If we narrow similarly a second set of integration limits, this is further reduced to $\frac{\pi^4}{26880} \approx 0.00362385$. An attempt to add a third set of similar integration limits — corresponding to a 3×3 minor — did not succeed computationally.) Applying formula (7.7) of the Życzkowski-Sommers study [5], we obtain for the HS-volume of the 4×4 real density matrices, $\frac{\pi^4}{60480} \approx 0.0016106$. This is $\frac{\pi^2}{54} \approx 0.18277$ times *smaller* than our first, principal calculation ($\frac{\pi^2}{1120}$), so we have a considerable overestimation.

If we *additionally* imposed the condition that the six principal minors of the partial transpose also have nonnegative determinants (only two of them being actually different from the original six), the result was $\frac{544}{99225} \approx 0.00548249$. (Note that $1120 = 2^5 \cdot 5 \cdot 7$ and $99225 = 3^4 \cdot 5^2 \cdot 7^2$.) So (taking the ratio) of this to $\frac{\pi^2}{1120}$, we obtain a crude estimate of the HS-separability probability of the real density matrices is 0.622151.

Unfortunately, our upper bound (0.00548249) on the HS-volume of the separable real two-qubit states is *larger* than the (known) HS-volume (0.0016106) of the (separable *and* nonseparable) two-qubit states, so we have not yet succeeded in deriving a nontrivial upper bound on the separable volume. (The same will be the case in the immediate next analysis.)

2. 15-Dimensional Complex Case

Now, when we again required that the six principal 2×2 minors have nonnegative determinants, we obtained for the corresponding Hilbert-Schmidt volume $\frac{\pi^6}{7882875} \approx 0.000121959$. (Note that $7882875 = 3^2 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$.) Formula (4.5) of [5] gives us for the HS-volume of the 15-dimensional convex set of two-qubit density matrices, the value $\frac{\pi^6}{85130500} \approx 1.12925 \cdot 10^{-6}$. (The ratio of these two volumes — the measure of our overestimation — is $\frac{7484}{693} \approx 10.7994$.) Imposing (just as we did in the *real* 9-dimensional case) the *further* requirements that the six 2×2 minors of the partial transpose all have nonnegative determinants, we obtain a HS-volume of $\frac{1964\pi^6}{30435780375} \approx 0.0000620378$. (Observe that $30435780375 = 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2$.) So, our crude separability *probability* estimate (less than in the 9-dimensional real case — as conforms with our intuition) is $\frac{1964}{3861} \approx 0.508677$. Based on certain numerical and theoretical considerations, the actual value of this (15-dimensional) separability probability has been conjectured to be [6, eq. (43)]

$$\frac{2^2 \cdot 3 \cdot 7^2 \cdot 11 \cdot 13\sqrt{3}}{5^3\pi^6} \approx 0.242379. \tag{30}$$

VIII. CONCLUDING REMARKS

We have found the newly-introduced capability of Mathematica (version 5.1) for integration over implicitly defined regions particularly useful for obtaining a very wide variety of separability (and positive-PPT) probabilities, particularly for low-dimensional ($m = 2, 3$) cases, essentially independently of the sizes (n) of the corresponding $n \times n$ density matrices analyzed. The use of such methods for cases $m >= 4$ appears, however — such as the two-qubit ($n = 4$) scenarios for the real ($m = 9$) and complex ($m = 15$) cases — to be particularly challenging.

Eggeling and Werner [13] studied the separability properties in a five-dimensional set of states of quantum systems composed of *three* subsystems of equal but arbitrary finite Hilbert space dimension. They are the states that commute with unitaries of the form $U \otimes U \otimes U$. In [14], we evaluated the probabilities of an Eggeling-Werner state being biseparable, triseparable or having a positive partial transpose with respect to certain partitions. However, the Hilbert-Schmidt measure was not employed, but rather the *Bures* one [15].

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